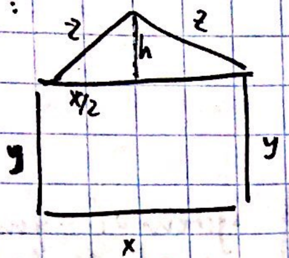
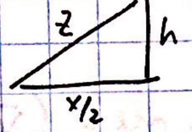
WH #3:





$\left(\frac{x}{2}\right)^2 + h^2 = z^2$

$h = \sqrt{z^2 - x^2/4}$

George WH #3.
Owen WH #1
Rujuta WH #4., W.4. PC1.
chi
Lizzy.

$A_{rect} = xy$

$A_{\text{"roof"}} = 2 \cdot A_{\triangle les} = \cancel{2} \cdot \frac{1}{\cancel{2}}\left(\frac{x}{2}\right)(h) = \frac{x}{2}\sqrt{z^2 - x^2/4}$

(2 $\triangle$les) $\frac{1}{2} \cdot b \cdot h$

$= \frac{x}{2}\sqrt{\frac{1}{4}(4z^2 - x^2)}$

$= \frac{x}{4}\sqrt{(4z^2 - x^2)}$

$\boxed{A_{total} = xy + \frac{x}{4}\sqrt{\dots}}$

want to opt.

$P = (\text{fixed}) = \underbrace{x + 2y + 2z}_{g}$

constraint

$\frac{x}{4}\sqrt{4z^2 - x^2}$

$\frac{x}{4}\frac{1}{2}\frac{1}{\sqrt{4z^2-x^2}} \cdot 8z$

$= \frac{zx}{\sqrt{4z^2 - x^2}}$

Lagrange: $\nabla A_{total} = \lambda \nabla g$

$$\begin{array}{c} \text{(i)} \\ \text{(ii)} \\ \text{(iii)} \end{array} \begin{bmatrix} y + \frac{1}{8}\frac{1}{\sqrt{\dots}}(8xz^2 - 4x^3) \\ x \\ \frac{1}{8}\frac{1}{\sqrt{4x^2z^2 - x^4}}(8x^2z) = \frac{xz}{\sqrt{4z^2 - x^2}} \end{bmatrix} = \begin{bmatrix} 1\lambda \\ 2\lambda \\ 2\lambda \end{bmatrix}.$$

(ii) $\lambda = x/2 \rightsquigarrow x = 2\lambda$

(iii) $\cancel{\frac{xz}{\sqrt{4z^2-x^2}}} \neq 1x$ $\quad \lambda = \frac{1}{2}\frac{xz}{\sqrt{4z^2 - x^2}}$

$\lambda = \frac{1}{2}\frac{(2\lambda)z}{\sqrt{4z^2 - 4\lambda^2}}$

$\sqrt{4z^2 - 4\lambda^2} = z$

$4z^2 - 4\lambda^2 = z^2$

$3z^2 = 4\lambda^2$

$z = \pm\frac{2}{\sqrt{3}}\lambda$

since $z > 0$, we know $z = \frac{2}{\sqrt{3}}\lambda$

~~$\lambda - \frac{1}{8}\sqrt{4z^2 - x^2}\ 8z^2 - 4x^2$~~

$\frac{d}{dx}\left(\frac{x}{4}\sqrt{4z^2 - x^2}\right) = \frac{1}{4}\sqrt{\dots} + \frac{x}{4}(4z^2 - x^2)^{-1/2} \cdot \frac{1}{2} \cdot (-2x)$

$= \frac{1}{4}\sqrt{\dots} - \frac{x^2}{2}\frac{1}{\sqrt{\dots}}$

$\sqrt{4z^2 - x^2}$.

(i) $y = \lambda - \frac{1}{8}\frac{1}{\sqrt{4x^2z^2 - x^4}}(8xz^2 - 4x^3)$

$= \lambda - \frac{1}{8}(8(2\lambda)$

$(\vdots)$ verify.

$\boxed{y = \frac{3 + \sqrt{3}}{3}\lambda}$

$A_{total}(x, y, z) \rightsquigarrow A_{tot}(\lambda) \rightsquigarrow \boxed{A_{tot}(P)}$

plug in
$x = 2\lambda$
$z = \dots$
$y = \dots$

$P = \dots(\lambda)$ $\left(\begin{smallmatrix}\text{"} \\ \vdots\end{smallmatrix}\right) P = \lambda$

$\hookrightarrow \boxed{A_{total} = (\dots)P}$

WH #1 $f(x,y) = x^2+y^2+kxy$ ... where does this change "qualitatively"?

Study the critical pts... $\left(\text{look @ } k \begin{cases} <-2 \\ \text{between } -2, \,\&\, 2 \\ >2 \end{cases}\right)$

$(\nabla f = 0.)$

punchline

WH #4 $f(x,y,z) = xyz$ $\quad x^2+y^2+z^2 \le 1$

$(x,y,z) = (\frac{1}{2},\frac{1}{2},\frac{1}{2})$

$\nabla f = \lambda \nabla g$

$$\begin{pmatrix} yz \\ xz \\ xy \end{pmatrix} = \begin{pmatrix} 2\lambda x \\ 2\lambda y \\ 2\lambda z \end{pmatrix} \underset{\text{(solve for } \lambda)}{\rightsquigarrow} \lambda = \frac{1}{2}\frac{yz}{x} = \frac{1}{2}\frac{xz}{y} = \frac{1}{2}\frac{xy}{z} \underset{\text{(solved for } y)}{\rightsquigarrow} \begin{array}{l} x(y) = \pm y \\ z(y) = \pm y \end{array}$$

$$2\lambda = \frac{yz}{x} = \frac{xz}{y} = \frac{xy}{z}.$$

$$\frac{yz}{x} = \frac{xz}{y} \rightsquigarrow x^2 = y^2 \rightsquigarrow x = \pm y.$$

$(z \neq 0)$

(i) $z=0$: can't be max ✓ — $f(x,y,0) = xy(0) = 0$.
(ii) $z \neq 0$.

but we know this can't be the max since $f(\frac{1}{2},\frac{1}{2},\frac{1}{2}) = \frac{1}{8} > 0$.

WH #5. $\frac{xy(x^2-y^2)}{(x^2+y^2)}$

$$\frac{\partial^2 f}{\partial x \partial y} = \begin{cases} \cancel{\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)} \\ \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \end{cases}$$

$$\frac{\partial f}{\partial x} = \frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2} \overset{\frac{\partial}{\partial y}}{\rightsquigarrow} \frac{-y^6-9y^4x^2+9y^2x^4+x^6}{(x^2+y^2)^3}$$

$$\frac{\partial f}{\partial y} = \frac{x(-y^4-4x^2y^2+x^4)}{(x^2+y^2)^2} \underset{\partial/\partial x}{\rightsquigarrow} \frac{+x^6+9x^4y^2-9y^2x^4-y^6}{(x^2+y^2)^3}$$

lims. same ✗.

Maybe we can use the limit defin of (partial) derivatives.

(1d): $\frac{df}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$ $\qquad$ (partial version:) $\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y)-f(x,y)}{h}$

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{1}{h}\left[\frac{(x+h)y((x+h)^2-y^2)}{((x+h)^2+y^2)} - \frac{xy(x^2-y^2)}{(x^2+y^2)}\right].$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{(x+h)y((x+h)^2-y^2)(x^2+y^2) - ((x+h)^2+y^2)xy(x^2-y^2)}{(x^2+y^2)((x+h)^2+y^2)}\right]$$

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$$\left.\frac{\partial^2 f}{\partial y \partial x}\right|_{(0,0)} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2}\right)\Bigg|_{(0,0)}$$

$$= \frac{\partial}{\partial y}(f_x)$$

$$= \lim_{h\to 0} \frac{f_x(0+h, 0) - f_x(0,0)}{h}$$

$f_x(0,0) \to 0$ (from (b))

$$= \lim_{h\to 0}$$

$$f_x(h,0) = \frac{h(h^4+4h^4-h^4)}{(2h^2)^2} = \frac{h(4h^4)}{4h^4} = h.$$

$f_x(0,0) \to 0$ (from (b))

$$= \lim_{h\to 0} \frac{f_x(0, 0+h) - f_x(0,0)}{h}$$

(evaluating $\frac{\partial}{\partial y}$ @ $(0,0)$ via limit def'n.)

$$f_x(0,h) = \frac{h(0+0-h^4)}{(0+h^2)^2} = \frac{-h^5}{h^4} = -h$$

$$= \lim_{h\to 0} \frac{-h}{h}$$

$$= \lim_{h\to 0}(-1) = -1.$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(f_y) = \dots = 1$$

$$\{ f_y(h,0) = \dots = h \}$$

what went wrong?
- continuity
- differentiability
- regularity.

Progress check #2 (14.4):

$f: \mathbb{R}^3 \to \mathbb{R}$

$(x,y,z) \mapsto (x^2+y^2+z^2) = f(x,y,z)$

$W = \{\underbrace{x^2+2y^2+z^2}_{g(x,y,z)} \le 1\}$. $\nabla f = \lambda \nabla g$.

$$\nabla f = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}. \qquad \nabla g = \begin{bmatrix} 2x \\ 4y \\ 2z \end{bmatrix}.$$

(i) $2x = \lambda(2x) \xrightarrow{(x\neq 0)} \lambda = 1$

(ii) $2y = \lambda(4y) \rightsquigarrow 2y = 4y \rightsquigarrow (y=0)$

(iii) $2z = \lambda(2z)$

$\boxed{x^2+z^2 \le 1}$

$f(x,0,z) = x^2+z^2$

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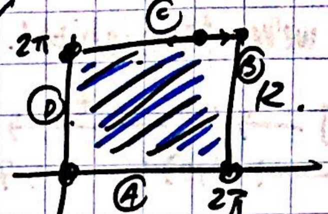
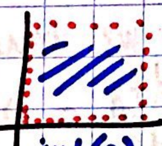
14.3 #7 : $f(x,y) = 7\sin(x) + 9\cos(y)$

on rectangle $R = \begin{Bmatrix} 0 \leq x \leq 2\pi \\ 0 \leq y \leq 2\pi \end{Bmatrix}$.



Want : abs. max/mins.

① find critical points ($\nabla f = 0$) → find (x,y) on the interior



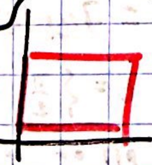
int(R) = $\begin{Bmatrix} 0 < x < 2\pi \\ 0 < y < 2\pi \end{Bmatrix}$

② find critical points on the boundary

(i) [redacted] break up the bdy (if necessary) into smooth pieces.

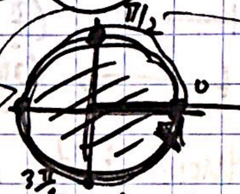
(ii) parametrize each piece.

turns into a function of 1-variable.



bdy(R) = $\begin{Bmatrix} x = 0, x = 2\pi \\ y = 0, y = 2\pi \end{Bmatrix}$.

(iii) solve as in Calc I. $\left(\frac{d}{dt} f\right)$



$c(t) = (\sin t, \cos t)$
$[0, 2\pi]$

③ Calculate what values the crit. pts get you & identify the biggest/smallest.

---

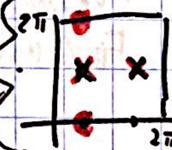
① find pts in interior ... $\begin{Bmatrix} 0 < x < 2\pi \\ 0 < y < 2\pi \end{Bmatrix}$ $\nabla f = 0$

$$\nabla f = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} 7\cos(x) \\ -9\sin(y) \end{bmatrix} = 0 .$$

(i) $7\cos(x) = 0 \rightsquigarrow x = \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$.

(ii) $-9\sin(y) = 0 . \rightsquigarrow y = \{\cancel{0}, \pi, \cancel{2\pi}\}$.

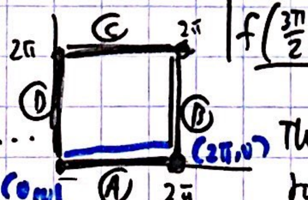
So our crit. pts inside are ... $(x,y) = \left\{\left(\frac{\pi}{2}, \pi\right), \left(\frac{3\pi}{2}, \pi\right)\right\}$



might as well find values ... $\boxed{f\left(\frac{\pi}{2}, \pi\right) = 7\sin\left(\frac{\pi}{2}\right) + 9\cos(\pi) = 7 + (-9) = -2}$

$\boxed{f\left(\frac{3\pi}{2}, \pi\right) = -7 - 9 = -16}$

② our boundary ...



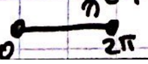
There are 4 lines (1-dim), so we should be able to describe them by a function

Parametrize $[0, 2\pi] \longrightarrow \mathbb{R}^2$
$\cap$
$\mathbb{R}$

e.g. Ⓐ we can parametrize

$[0, 2\pi] \longrightarrow \mathbb{R}^2$

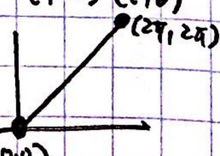
$t \longmapsto (t, 0)$



Ⓑ $[0, 2\pi] \longrightarrow \mathbb{R}^2$

$t \longmapsto (2\pi, t)$

e.g. $[0, 2\pi]$
$t \longmapsto (t, t)$



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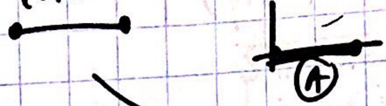
(ii) once we've parametrized...

$$\begin{tikzcd} {[0,2\pi]} \arrow[r, "A"] \arrow[rr, bend right] & \mathbb{R}^2 \arrow[r, "f"] & \mathbb{R}. \end{tikzcd}$$



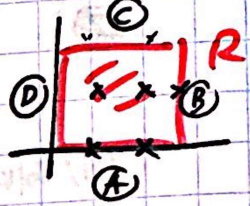
$$\begin{tikzcd} {[0,2\pi]} \arrow[r] & \mathbb{R}^2 \arrow[r, "f"] & \mathbb{R} \\ t \arrow[r, mapsto] \arrow[rr, bend right, "F_A"'] & (t,0) \arrow[r, mapsto] & 7\sin(t)+9\cos(0) \end{tikzcd}$$

function of one-variable.

$$[0,2\pi] \xrightarrow[F_A]{} \mathbb{R}.$$

Now we want to find crit. pts of a 1-dim. function.. (calc 1)

$$\frac{dF_A}{dt} = 0. \qquad \frac{dF_A}{dt} = \frac{d}{dt}\left(7\sin(t) + 9\cos(0)\right)$$
$$= 7\cos(t) \qquad = 0.$$
$$\rightsquigarrow t = \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}.$$



(iii) We're interested in crit. pts of f (ie. want pts in the plane)

$$\begin{aligned} {[0,2\pi]} &\xrightarrow{A} \mathbb{R}^2 \\ t &\longmapsto (t,0) \\ \frac{\pi}{2} &\longmapsto \left(\frac{\pi}{2},0\right) \\ \frac{3\pi}{2} &\longmapsto \left(\frac{3\pi}{2},0\right). \end{aligned}$$

Go through & do this also for B, C, D.

(check the corners).

3 We've got a collection of crit. pts $= \left\{ \begin{matrix} \left(\frac{\pi}{2},\pi\right), & \left(\frac{3\pi}{2},\pi\right) \\ \left(\frac{\pi}{2},0\right), & \left(\frac{3\pi}{2},0\right) \\ \vdots & \end{matrix} \right\}.$

Evaluate each (plugging into f).
Find which are the biggest/smallest.