

Practice quiz #4.

$$f(u,v) = (\tan(u-1) - e^v, 5u^2 - 4v^2)$$

$$g(x,y) = (e^{6(x-y)}, 6(x-y))$$

$f \circ g = ?$

Q: What should this "eat" & "spit out?"

f & g are both functions $\mathbb{R}^2 \rightarrow \mathbb{R}^2$... what does $f \circ g$ represent?

(i) $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}^2 \xrightarrow{g} \mathbb{R}^2$
 $(a,b) \mapsto (f_1, f_2) \mapsto (g_1(f_1), g_2(f_2))$ X

(ii) $\mathbb{R}^2 \xrightarrow{g} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}^2$
 $(a,b) \mapsto (g_1, g_2) \mapsto (f_1(g_1), f_2(g_2))$ ✓
 $f \circ g$ (same order)

Let's calculate the thing...

$$\mathbb{R}^2 \xrightarrow{g} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}^2$$

$$(x,y) \mapsto (g_1(x,y), g_2(x,y)) \mapsto (\tan(u-1) - e^v, 5u^2 - 4v^2)$$

$$\left(\underbrace{e^{6(x-y)}}_u, \underbrace{6(x-y)}_v \right) \mapsto \left(\tan(u-1) - e^v, 5u^2 - 4v^2 \right)$$

$$\left\{ \begin{aligned} \tan(e^{6(x-y)} - 1) - e^{6(x-y)} & , \quad 5(e^{6(x-y)})^2 - 4(6(x-y))^2 \\ 5e^{12(x-y)} - 4(6^2)(x-y)^2 & \\ 5e^{12(x-y)} - 4(36)(x-y)^2 & \end{aligned} \right\} = f \circ g(x,y)$$

$D(f \circ g)(1,1) = ?$

What is $D(F)$

Some function $\mathbb{R}^m \xrightarrow{F} \mathbb{R}^n$
 $(x_1, \dots, x_m) \mapsto (F_1, \dots, F_n)$

$$D(F) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \dots & \frac{\partial F_1}{\partial x_m} \\ \frac{\partial F_2}{\partial x_1} & \dots & \dots & \dots \\ \vdots & \dots & \dots & \dots \\ \frac{\partial F_n}{\partial x_1} & \dots & \dots & \frac{\partial F_n}{\partial x_m} \end{bmatrix}$$

In general...

$$\mathbb{R}^2 \xrightarrow{h_1} \mathbb{R}^m \xrightarrow{h_2} \mathbb{R}^n$$

$$D(h_2 \circ h_1) = Dh_2 \cdot Dh_1$$

In our case, $F = (f \circ g): \mathbb{R}^2 \rightarrow \mathbb{R}^2$, (so $m=2, n=2$)
 so $DF = D(f \circ g)$ should be a 2×2 matrix.

$$DF = D(f \circ g) = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{bmatrix}$$

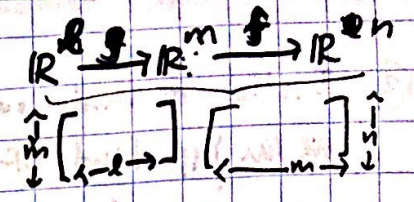
where $F_i =$

$$n \begin{bmatrix} \] = n \begin{bmatrix} \] \begin{bmatrix} \]$$

$$= Df \cdot Dg \Big|_{(1,1)} = [2 \times 2] [2 \times 2] = [2 \times 2]$$

D of chain rule...

$$D(f \circ g)|_{(x_0, y_0)} = Df|_{g(x_0, y_0)} \cdot Dg|_{(x_0, y_0)}$$



In our case... $(x_0, y_0) = (1, 1)$

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{bmatrix}$$

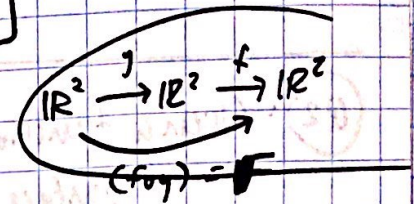
$$Dg = \begin{bmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{bmatrix}$$

$$f_1 = \tan(u-1) - e^v$$

$$f_2 = 5u^2 - 4v^2$$

$$g_1 = e^{6(x-y)}$$

$$g_2 = 6(x-y)$$



$$\frac{\partial f_1}{\partial u} = \frac{\partial}{\partial u}(\tan(u-1) - e^v)$$

$$= \frac{1}{\cos^2(u-1)}$$

$$\frac{\partial f_1}{\partial v} = \frac{\partial}{\partial v}(\tan(u-1) - e^v)$$

$$= -e^v$$

$$\frac{\partial g_1}{\partial x} = e^{6(x-y)} \cdot (6)$$

$$\frac{\partial g_1}{\partial y} = e^{6(x-y)} \cdot (-6)$$

$$\frac{\partial g_2}{\partial x} = 6$$

$$\frac{\partial g_2}{\partial y} = -6$$

$$\frac{\partial f_2}{\partial u} = 10u$$

$$\frac{\partial f_2}{\partial v} = -8v$$

$$Df = \begin{bmatrix} \frac{1}{\cos^2(u-1)} & -e^v \\ 10u & -8v \end{bmatrix}$$

$$Dg = \begin{bmatrix} 6e^{6(x-y)} & -6e^{6(x-y)} \\ 6 & -6 \end{bmatrix}$$

plug in @ the relevant points...

$$Df|_{g(1,1)} = \begin{bmatrix} \frac{1}{\cos^2(1-1)} & -e^0 \\ 10(1) & -8(0) \end{bmatrix}$$

$$Dg|_{(1,1)} = \begin{bmatrix} 6e^{6(1-1)} & -6e^{6(1-1)} \\ 6 & -6 \end{bmatrix}$$

$$g(1,1) = \langle e^{6(1-1)}, 6(1-1) \rangle = \langle 1, 0 \rangle$$

$$= \begin{bmatrix} 1 & -1 \\ 10 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ 10 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -6 \\ 6 & -6 \end{bmatrix}$$

multiply matrices... $D(f \circ g)|_{(1,1)} = Df|_{g(1,1)} \cdot Dg|_{(1,1)}$

$$= \begin{bmatrix} 1 & -1 \\ 10 & 0 \end{bmatrix} \begin{bmatrix} 6 & -6 \\ 6 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 60 & -60 \end{bmatrix}$$

②. Instead of chain rule, since we found $F = \nabla f$ already...
 we can just find $Df \dots$

$$F(x,y) = \left\langle \underbrace{\tan(e^{6(x-y)} - 1) - e^{6(x-y)}}_{F_1}, 5e^{12(x-y)} - 144(xy)^2 \right\rangle \dots$$

Q2: Given a function $f(x,y,z) = 3xyz$.

surface $yx^2 + xy^2 + yz^2 = 4$.

$g(x,y,z)$ @ (1,1,1)

to find the rate of change of f in some direction \vec{v}

$\nabla f|_{(1,1,1)} \cdot \vec{u}$

dot product

unit vector
 $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}$!

In our problem, we want \vec{v} to be the direction normal to the surface...
 $\vec{v} = \nabla g|_{(1,1,1)}$

