Practice quiz 3 #4.

$f(u,v) = (\tan(u-1) - e^{v},\ 5u^2 - 4v^2)$

$g(x,y) = (e^{6(x-y)},\ 6(x-y))$

A function $F: \mathbb{R}^m \to \mathbb{R}^n$

$(x_1 \ldots x_m) \mapsto (F_1, F_2 \ldots F_n)$

$F_1(x_1, x_2 \ldots x_m)$

$\boxed{f \circ g = ?}$

Q: What should this "eat" & "spit out"?
f & g are both functions $\mathbb{R}^2 \to \mathbb{R}^2$ ... what does $f \circ g$ represent?

(i) $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}^2 \xrightarrow{g} \mathbb{R}^2$
$(a,b) \mapsto (f_1, f_2) \mapsto (g_1(f_1), g_2(f_2))$ ✗

(ii) $\mathbb{R}^2 \xrightarrow{g} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}^2$
$(a,b) \mapsto (g_1, g_2) \mapsto (f_1(g_1), f_2(g_2))$ ✓.

$f \circ g$ (same order)

Let's calculate the thing...

$\mathbb{R}^2 \xrightarrow{g} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}^2$

$(x,y) \mapsto (g_1(x,y), g_2(x,y))$

$\langle \underbrace{e^{6(x-y)}}_{u}, \underbrace{6(x-y)}_{v} \rangle \mapsto \langle \tan(u-1) - e^{v}, 5u^2 - 4v^2 \rangle$

$\langle \tan(e^{6(x-y)} - 1) - e^{6(x-y)},\ 5(e^{6(x-y)})^2 - 4(6(x-y))^2 \rangle$

$= 5e^{12(x-y)} - 4(6^2)(x-y)^2$

$\langle \quad , \quad 5e^{12(x-y)} - 4(36)(x-y)^2 \rangle = f \circ g\,(x,y)$

$\boxed{D(f \circ g)(1,1) = ?}$

What is $D(F)$

some function $\mathbb{R}^m \xrightarrow{F} \mathbb{R}^n$

$(x_1 \ldots x_m) \mapsto (F_1 \ldots F_n)$

$$D(F) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_m} \\ \frac{\partial F_2}{\partial x_1} & \ddots & & \\ \vdots & & & \\ \frac{\partial F_n}{\partial x_m} & & & \frac{\partial F_n}{\partial x_m} \end{bmatrix}$$

In general...

$\mathbb{R}^l \xrightarrow{h_1} \mathbb{R}^m \xrightarrow{h_2} \mathbb{R}^n$

$D(h_2 \circ h_1) = Dh_2 \cdot Dh_1$

$n\{\underbrace{[\quad]}_{l} = n\{\underbrace{[\quad]}_{m}\ m\{\underbrace{[\quad]}_{l}$

In our case, $F = (f \circ g) : \mathbb{R}^2 \to \mathbb{R}^2$, (so $m=2$, $n=2$)
so $DF = D(f \circ g)$ should be a 2×2 matrix.

where $F_1 =$

$n[\quad]_l = n[\quad]_m\ m[\quad]_l$.

$$DF = D(f \circ g) = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{bmatrix}$$

$= Df\big|_{(g(1,1))} \cdot Dg\big|_{(1,1)} = [2\times 2][2\times 2] = [2 \times 2]$

D w/ chain rule... $D(f\circ g)\big|_{(x_0,y_0)} = Df\big|_{(g(x_0,y_0))} \cdot Dg\big|_{(x_0,y_0)}$ F !

$\mathbb{R}^{l} \xrightarrow{g} \mathbb{R}^{m} \xrightarrow{f} \mathbb{R}^{n}$

$m \updownarrow \begin{bmatrix} \longleftarrow l \longrightarrow \end{bmatrix} \begin{bmatrix} \longleftarrow m \longrightarrow \end{bmatrix} \updownarrow n$

① In our case... $(x_0,y_0) = (1,1)$

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{bmatrix} \qquad Dg = \begin{bmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{bmatrix}$$

$\mathbb{R}^2 \xrightarrow{g} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}^2$, $(f\circ g) = F$

$f_1 = \tan(u-1) - e^v$

$f_2 = 5u^2 - 4v^2$

$g_1 = e^{6(x-y)}$

$g_2 = 6(x-y)$

$$\frac{\partial f_1}{\partial u} = \frac{\partial}{\partial u}(\tan(u-1) - e^v) = \frac{1}{\cos^2(u-1)} \qquad \frac{\partial f_1}{\partial v} = \frac{\partial}{\partial v}(\tan(u-1) - e^v) = -e^v$$

$$\frac{\partial g_1}{\partial x} = e^{6(x-y)}\cdot(6) \qquad \frac{\partial g_1}{\partial y} = e^{6(x-y)}(-6)$$

$$\frac{\partial g_2}{\partial x} = 6 \qquad \frac{\partial g_2}{\partial y} = -6.$$

$$\frac{\partial f_2}{\partial u} = 10u \qquad \frac{\partial f_2}{\partial v} = -8v$$

$$Df = \begin{bmatrix} \frac{1}{\cos^2(u-1)} & -e^v \\ 10u & -8v \end{bmatrix} \qquad Dg = \begin{bmatrix} 6e^{6(x-y)} & -6e^{6(x-y)} \\ 6 & -6 \end{bmatrix}$$

$\to$ plug in @ the relevant points..

$$Df\big|_{g(1,1) = \langle 1,0\rangle} = \begin{bmatrix} \frac{1}{\cos^2(1-1)} & -e^0 \\ 10(1) & -8(0) \end{bmatrix} = \begin{bmatrix} \frac{1}{1} & -1 \\ 10 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 10 & 0 \end{bmatrix}.$$

$$g(1,1) = \langle e^{6(1-1)}, 6(1-1)\rangle = \langle 1, 0\rangle$$

!

$$Dg\big|_{(1,1)} = \begin{bmatrix} 6e^{6(1-1)} & -6e^{6(1-1)} \\ 6 & -6 \end{bmatrix} = \begin{bmatrix} 6 & -6 \\ 6 & -6 \end{bmatrix}.$$

$\hookrightarrow$ multiply matrices.. $D(f\circ g)\big|_{(1,1)} = Df\big|_{g(1,1)} \cdot Dg\big|_{(1,1)}$

$$= \begin{bmatrix} 1 & -1 \\ 10 & 0 \end{bmatrix} \begin{bmatrix} 6 & -6 \\ 6 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 60 & -60 \end{bmatrix}.$$

②. Instead of chain rule, since we found $F = f \circ g$ already...

we can just find DF...

$$F(x,y) = \Big\langle \underbrace{\tan\left(e^{6(x-y)}-1\right) - e^{6(x-y)}}_{F_1},\ 5e^{12(x-y)} - 144(x-y)^2 \Big\rangle.$$

Q2: Given a function $f(x,y,z) = 3xyz$.

surface $\underbrace{yx^2 + xy^2 + yz^2}_{g(x,y,z)} = 4$.

@ $(1,1,1)$

to find the rate of change of f_1 in some <u>direction</u> $\cdot\ \vec{v}$

In our problem, we want $\vec{v}$ to be the direction normal to the surface...

$$\nabla f\Big|_{(1,1,1)} \cdot \vec{u}$$

dot product

unit vector $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|}$ (!)

$$\vec{v} = \nabla g\Big|_{(1,1,1)}$$