$f: U \to \mathbb{R}$

$U \subset \mathbb{R}^3$

U is some region sitting in $\mathbb{R}^3$

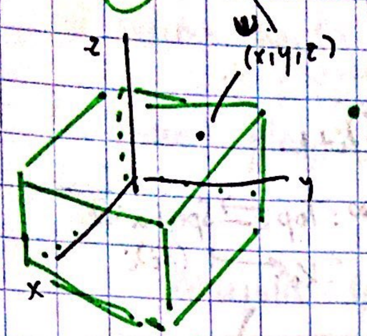
3-d. space

ie. x, y, z space.

$f: (x,y,z) \mapsto f(x,y,z)$

"sends the point to..."

tells you that f spits out a $\mathbb{R}$ (real #).



$x \in U$

x is a pt in U

$U \subseteq \mathbb{R}^3$

U is a subset of $\mathbb{R}^3$

$$\left.\begin{cases} 0 < x < 5 \\ 0 < y < 10 \\ 0 < z < 10 \end{cases}\right\} U$$

$f: U \to \mathbb{R}^3$

$U \subset \mathbb{R}^5$

$(x_1, x_2, x_3, x_4, x_5) \mapsto f(x_1, \ldots x_5)$

$= (f_1(\ldots), f_2(\ldots), f_3(\ldots))$

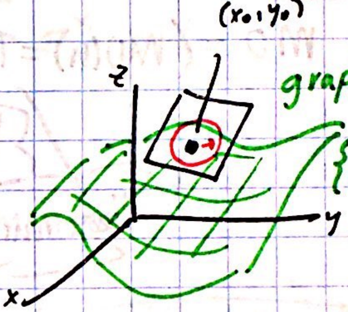
Df

		videos.
(i) tangent plane. (2-variables)		(13.3.4)
(ii) Df	(multivariable)	(13.3.5)
(iii) differentiability (limit)		(13.3.6)

(13.3.4) functions of 2-variables.

$$\begin{cases} f: \mathbb{R}^2 \to \mathbb{R} \\ f: U \to \mathbb{R}, \quad U \subset \mathbb{R}^2 \end{cases}$$

$(x,y) \mapsto f(x,y) = "z"$



In this graphs will look like surfaces in $\mathbb{R}^3$.

We learned how to find tangent planes

ie.

- linear (straight, easy to look @)
- approximate your graph/function.

ie. linear approximations.

a pt. on the tang. plane near (x_0, y_0)

a pt. on the graph near (x_0, y_0)

close together, b/c the tangent plane approximates the function (near (x_0, y_0)).

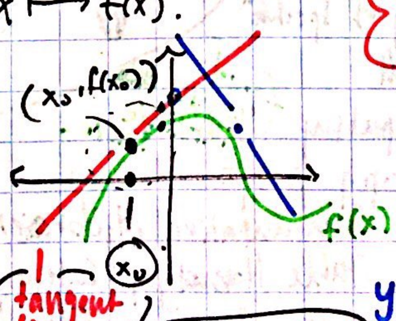
Remember... in calc I..

We would take a function $f: \mathbb{R}^1 \to \mathbb{R}^1$

$x \mapsto f(x)$.

graph looked like curves in $\mathbb{R}^2$

$\text{graph}(f) = \{(x, \underset{y}{f(x)})\} \subseteq \mathbb{R}^2$.



Instead of tangent planes (via partials $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$)

We took tangent lines (via reg. derivatives $\frac{df}{dx}$)

tangent line

$y = m\,x + b.$

$$y = \left.\frac{df}{dx}\right|_{x_0} (x - x_0) + f(x_0) = \left(\left.\frac{df}{dx}\right|_{x_0}\right) x + \left(f(x_0) - \frac{df}{dx}(x_0)\right)$$

ex, compare this to the equation of a tangent plane.

(i) If $h = cf$ then $Dh = D(cf) = cDf$.

(ii) If $h = f+g$, then $D(h) = D(f+g) = Df + Dg$.

(iii) If $h = fg$, then

$$D(h) = D(fg) = (Df)g + f(Dg)$$

(iv) quotient?

treat D like $\frac{d}{dx}$ from normal calc.

$$\frac{d}{dx}(fg) = \frac{df}{dx}\cdot g + f\cdot\frac{dg}{dx}.$$

ex//. $f(x,y,z) = xy + e^z$ $((..),(..))$

$g(x,y,z) = x^2 + y^2$

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \ddots & & \\ \vdots & & & \\ \frac{\partial f_m}{\partial x_1} & \cdots & & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

dis appears.. (no $f_2, f_3, \ldots$)

$$D(fg) = (Df)g + f(Dg).$$

In our case, f spits out a real #,

so $f = f_1(x,y,z)$

so $Df = \left[\frac{\partial f_1}{\partial x} \quad \frac{\partial f}{\partial y} \quad \frac{\partial f}{\partial z}\right]$. $Dg = \left[\frac{\partial g}{\partial x} \quad \frac{\partial g}{\partial y} \quad \frac{\partial g}{\partial z}\right]$

$Df = \left[y \quad x \quad e^z\right]$ $Dg = \left[2x \quad 2y \quad 0\right]$

$$Df\cdot g = [y, x, e^z](x^2+y^2)$$

$$= \left[(x^2y+y^3), (x^3+y^2x), e^z(x^2+y^2)\right]$$

$$3\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3a & 3b \\ 3c & 3d \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} (a+w) & (b+x) \\ (c+y) & (d+z) \end{bmatrix}$$

$$f\cdot Dg = (xy+e^z)\left[2x, 2y, 0\right]$$

$$= \left[(2x^2y + 2xe^z), (2y^2x + 2ye^z), 0\right]$$

- tangent planes 13.7.4.
- differentiability.
- HW question 13.4.
- HW. 13.3.

$Df = ?$.
"total derivative".

$f: \mathbb{R}^m \longrightarrow \mathbb{R}^n$

$(x_1 \cdots x_m) \longmapsto (f_1(..), f_2(..), \ldots, f_n(..))$

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \ddots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & & \frac{\partial f_m}{\partial x_n} \end{bmatrix}.$$

$\frac{\partial f_i}{\partial x_j}$ $\begin{pmatrix} i = 1,2,\ldots,n \\ j = 1,2,\ldots m. \end{pmatrix}$

(collect all these guys into a matrix..

There's a special Df..
when f is a real-valued function, ..

$f: \mathbb{R}^m \longrightarrow \mathbb{R}^1$

$(x_1 \cdots x_m) \longmapsto f(x_1 \cdots x_m)$

In this case,.

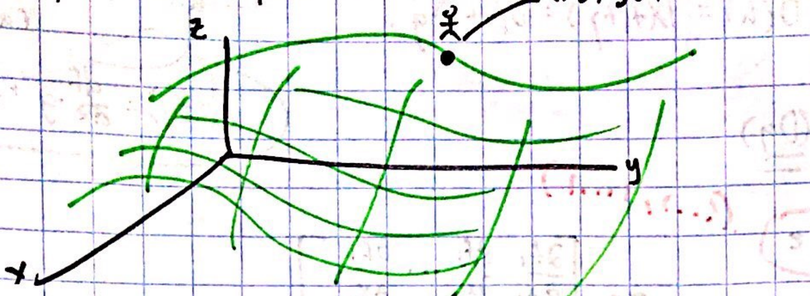
$Df = \left[\frac{\partial f}{\partial x_1} \cdots \frac{\partial f}{\partial x_m}\right]$. is a $(m\times 1)$-matrix (a row)

We call this guy the gradient $\nabla f = Df$ in this case.

- direction of fastest increase.
- normal vector of tangent plane

Say $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ ~~$f: \mathbb{R}^3 \Rightarrow \mathbb{R}$.~~

↳ graphs of this guy form surfaces.

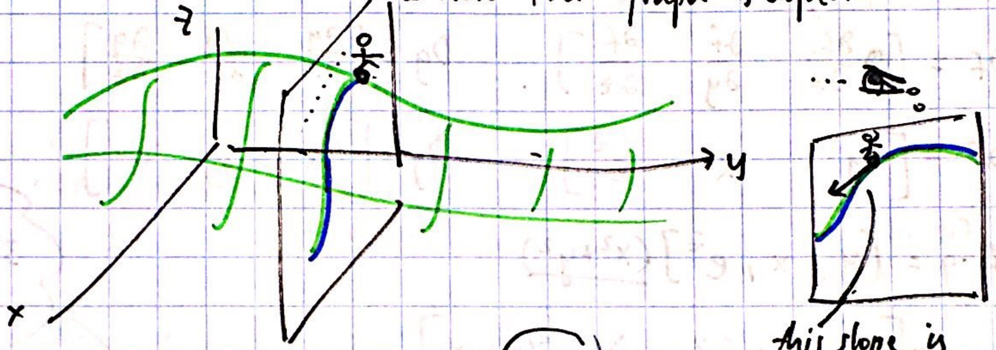


What does ∇f look like?

$$\nabla f = Df = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}. \quad (2\times 1) \; (1\times 2)? \text{ matrix.}$$

(0) Stand on the graph...

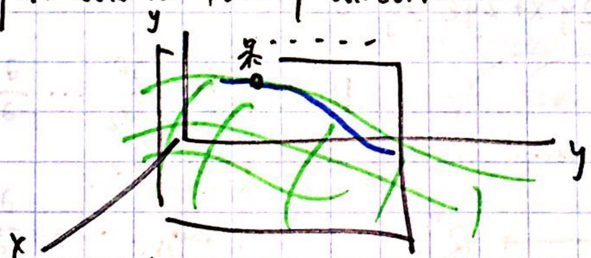
(1) Look in the x-direction & see how the function changes i.e. see how the graph is sloped.

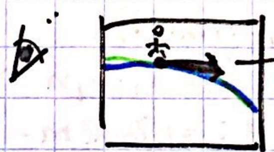


this slope is

$\frac{\partial f}{\partial x}$ — { "how f changes moving in the x-direction".

(2) Similarly... look in the y-direction..

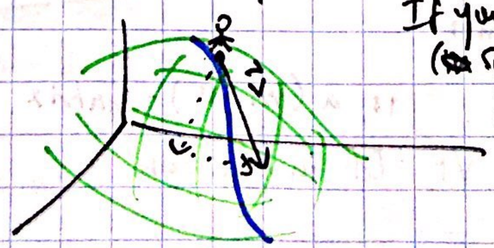




"how f changes in" the y-direction ~ $\frac{\partial f}{\partial y}$.

The gradient captures all this info at once.

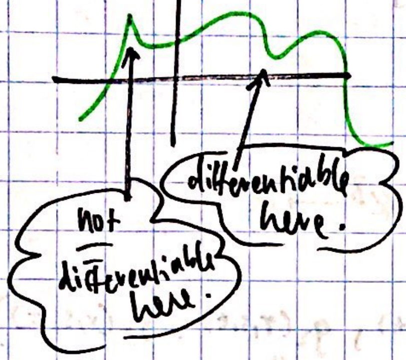
It turns out that this captures info. about any direction.



If you want to look in an arb. direction. (say $\vec{v}$) Since we can write $\vec{v}$ in terms of x & y components, we only need to know the $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ to find $\frac{\partial f}{\partial \vec{v}}$ — we'll see how to actually do this soon...

In Calc I ... we saw that a function $\mathbb{R}^1 \to \mathbb{R}^1$ was differentiable if it's "smooth". ($\mathbb{R}^n \to \mathbb{R}^n$)

imprecise



So we learned a more precise version involving limits...

f is differentiable (at x_0) if a certain limit exists... its derivative exists &

$\frac{df}{dx}$ exists

$$\cancel{\lim_{x \to x_0} \frac{\|f(x) - f(x_0)\|}{\|x - x_0\|} = \frac{df}{dx}\Big|_{x_0}}$$

$$\lim_{x \to x_0} \frac{\|f(x) - f(x_0)\|}{\|x - x_0\|} = \frac{df}{dx}\Big|_{x_0}$$

— or —

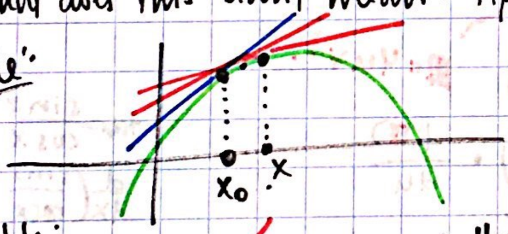
$$\lim_{x \to x_0} \frac{\|f(x) - f(x_0)\|}{\|x - x_0\|} - \frac{df}{dx}\Big|_{x_0} = 0$$

Calc I version of differentiability.

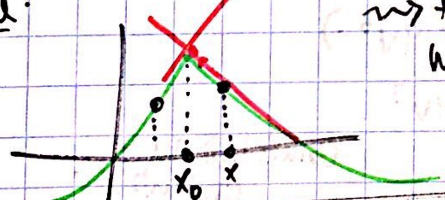
$$\boxed{\lim_{x \to x_0} \frac{\|f(x) - f(x_0) - \frac{df}{dx}\big|_{x_0}(x - x_0)\|}{\|x - x_0\|} = 0.}$$

Q: What does this limit mean? A: Approach x_0 & see how tangent slopes behave. $\frac{\|f(x) - f(x_0)\|}{\|x - x_0\|}$

diff'ble:



not-diff'ble:



⇝ these don't agree from different sides. Work out mathematically what this means, it means that the limit doesn't exist.

In vector calc... differentiability looks like ... $f: \mathbb{R}^n \to \mathbb{R}^m$ is diff'ble @ $\vec{x}_0$ if...

$$\boxed{\lim_{\vec{x} \to \vec{x}_0} \frac{\|f(\vec{x}) - f(\vec{x}_0) - Df\big|_{\vec{x}_0}(\vec{x} - \vec{x}_0)\|}{\|\vec{x} - \vec{x}_0\|} = 0.}$$

ex/ (compare)

in books they call it T

$T = Df\big|_{\vec{x}_0}$.

ex/ Look @ when f is real-valued... then

$Df = \nabla f$

& this looks a little nicer..

Given f,g … $g: \mathbb{R}^n \to \mathbb{R}^m$. $f: \mathbb{R}^m \to \mathbb{R}^p$

$$\mathbb{R}^n \xrightarrow{g} \mathbb{R}^m \xrightarrow{f} \mathbb{R}^p, \quad \text{with } f\circ g: \mathbb{R}^n \to \mathbb{R}^p.$$

$$D(f\circ g)(\vec{x}_0) = Df(g(\vec{x}_0)) \cdot Dg(\vec{x}_0)$$

$\vec{x}_0 \in \mathbb{R}^n$

matrix multiplication

In our case … $f(x,y,z) = xy + z^5$ $\quad \mathbb{R}^3 \to \mathbb{R}$

$$\begin{pmatrix} x = r+s-5t \\ y = 3rt \\ z = s^6 \end{pmatrix} \quad g: \mathbb{R}^3 \to \mathbb{R}^3$$

$(r,s,t) \mapsto (g_1(r,s,t),\ g_2(r,s,t),\ g_3(r,s,t))$

$g_1 = r+s-5t$, $g_2 = 3rt$, $g_3 = s^6$

$$\mathbb{R}^3 \xrightarrow{g} \mathbb{R}^3 \xrightarrow{f} \mathbb{R}^1.$$

<u>two matrices:</u>

(i) $Df(g(\vec{x}_0))$

(ii) $Dg(\vec{x}_0)$

Maybe an easier way to calculate this is …

$$\frac{\partial f}{\partial t} = \underbrace{\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t}} + \underbrace{\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}} + \underbrace{\frac{\partial f}{\partial z}\frac{\partial z}{\partial t}}.$$

$$\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} \qquad \frac{\partial f}{\partial x} = y = 3rt$$

$$\frac{\partial x}{\partial t} = \frac{\partial}{\partial t}(r+s-5t) = -5.$$

$-15rt$.

13.3 @ HW #12, 4.

(#4). $z = \tan(4uv^3)$ $\quad \square = 4uv^3$.

$$\frac{\partial z}{\partial u} = \underbrace{\frac{d}{d\square}(\tan(\square))} \cdot \frac{d(\square)}{du}$$

$$= \sec^2(\square) \cdot (4v^3)$$

$$= \sec^2(4uv^3)\cdot 4v^3.$$

$\tan = \dfrac{\sin x}{\cos x}$

$\dfrac{d}{dx}\left(\dfrac{\sin x}{\cos x}\right)$

$$\frac{\partial z}{\partial v} = \tan(4uv^3) \quad \text{(blackbox all constants)}.$$

$$= \sec^2(4uv^3)\cdot\frac{\partial}{\partial v}(4uv^3)$$

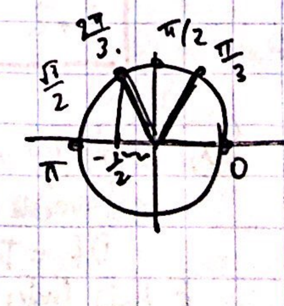
$$4u\frac{\partial}{\partial v}(v^3)$$

$$4u(3v^2)$$

(#12) . $G(u,w) = -\cos(uw)$

@ pt. $(\pi, 2/3) = (u_0, w_0)$

$$z = \underbrace{\frac{\partial G}{\partial u}\Big|_{(..)}(u-u_0)}_{(i)} + \underbrace{\frac{\partial G}{\partial w}\Big|_{(..)}(w-w_0)}_{(ii)} + \underbrace{G(u_0,w_0)}_{(iii)}$$



$-\cos\left(\frac{2\pi}{3}\right) = +\frac{1}{2}$

$$\frac{\partial G}{\partial u} = \frac{d}{d\square}(\cos(\square))\cdot\frac{\partial}{\partial u}(uw)$$

$$= \sin(uw)\cdot(w)$$

~~(i)~~ $\sin(uw)(w)(u-\pi)$