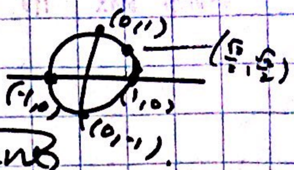
Sets in general are just a way of describing collections of things based on some property/properties... $\{\text{what the things look like} \mid : \text{properties they satisfy}\}$



eg. If we want to describe the unit circle in $\mathbb{R}^2$. — {xy-plane}

we can describe it as a set

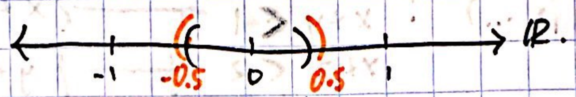
$$\{\underbrace{(x,y) \in \mathbb{R}^2}_{\text{all pts in plane}} : \underbrace{x^2+y^2=1}_{\text{throw out all points that don't satisfy this}}\}.$$

{keep only pts satisfying this}

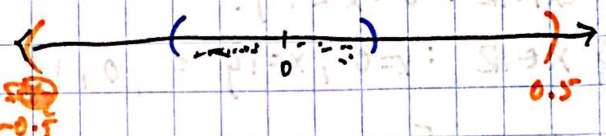
Pick $(2,3) \in \mathbb{R}^2$. Does this satisfy this relation?
$2^2+3^2 = 4+9 = 13 \neq 1$ NO. Throw it out.

Open sets (in $\mathbb{R}^n$)

Look at the real line $\mathbb{R}$.



"topology".



An open set U is a subset of $\mathbb{R}^n$ where any point inside U has a "neighborhood" that sits entirely inside U.

{a smaller open set}

{$\mathbb{R}^2$, $\mathbb{R}^3$}

in the book... any point in U has a tiny disk/ball sitting entirely inside U.

{$D_r(\vec{x}_0)$}

eg of disks in dim = n (n=2):

$\{(x,y) : x^2+y^2<1\}$.

(n=3):

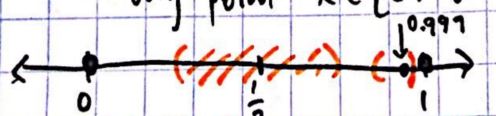
$\{(x,y,z) : x^2+y^2+z^2<1\}$.

(n=1):

$\{(x) : x^2 < 1\}$.
$|x| < 1$
$-1 < x < 1$

$D \subseteq [0,1]$

eg. Is $[0,1]$ open?
ie. any point $x \in [0,1]$ (ie. $0 \le x \le 1$) should have a disk entirely inside [0,1]

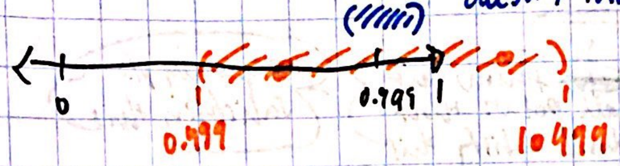


(zoom in)

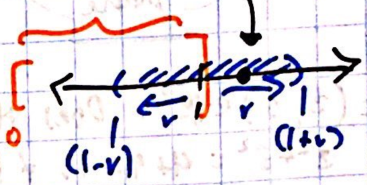
$D = (\frac{1}{4}, \frac{3}{4}) \subseteq [0,1]$ $(0.8, 0.9995) \subseteq [0,1]$

Take $(0,1)$.
What are its bdy points?
↳ a point where if you take any nbhd around it, (disk)
it includes · a point inside the U.
· a point outside the U.

e.g. (0.999). The disk of radius $\frac{1}{2}$.
doesn't include any pts outside.



But 1. Take any disk around 1.
$D_r(1) = \{ x \in \mathbb{R} \mid (1-r) < x < (1+r) \}$.



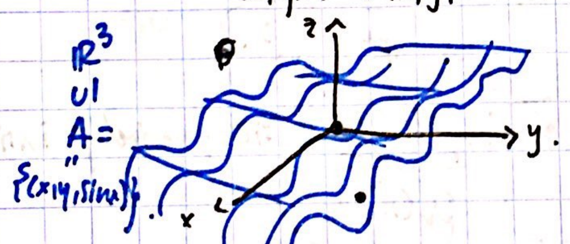
In general... Open sets may be defined as like.

$\{ (x,y,z \ldots) : x+y < 1 \quad y+z < 2 \}$ — $\{x+y=1\}$, $\{y+z=2\}$ bdy points.

e.g. $(0,1) = \{ (x) \in \mathbb{R} : 0 < x < 1 \}$.
bdy pts $= \{ x \in \mathbb{R} : x=0 , x=1 \} = \{0,1\}$.

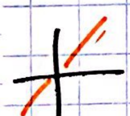
$\mathbb{R}^2 \xrightarrow{F} \mathbb{R}^3$
$(x,y) \longmapsto (x,y,\sin(x))$

$\mathbb{R}^3 \longrightarrow \mathbb{R}^2$
(x,y,z)



$f: A \to \mathbb{R}^2$
$(x,y,\sin x) \mapsto (x+\sin x, y+\sin x)$

$F: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$
$(x,y) \longmapsto (F_1(x,y), F_2(x,y), F_3(x,y))$
$F_1(x,y) = x$, $F_2(x,y) = y$, $F_3(x,y) = \sin(x)$



Given a real-valued function $f: \mathbb{R}^n \to \mathbb{R}$
Its graph
$\text{graph}(f) = \{ (x_1, x_2, \ldots, x_{n+1}) : x_{n+1} = f(x_1, \ldots x_n) \}$
$\cap$
$\mathbb{R}^{n+1}$

$= \{ \underbrace{(x_1, x_2, \ldots, x_n}, f(x_1 \ldots x_n)) \}$.

$L: \mathbb{R} \longrightarrow \mathbb{R}^2$
$x \longmapsto (L_1(x), L_2(x))$
$L_1(x) = x$, $L_2(x) = x$

$\{ (x, f(x)) \}$.

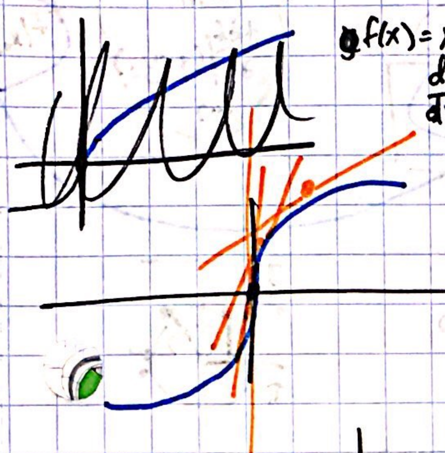
(1) What does it mean for a vector-valued function to be diff'ble?
(2) Linear approximation / tangent planes?

Def. A function $f: U \to \mathbb{R}^m$, $U \subseteq \mathbb{R}^n$, is $\boxed{\text{differentiable at } \vec{x}_0}$ if ...

(i) All partials exist. $\left.\dfrac{\partial f}{\partial x_i}\right|_{\vec{x}_0}$ $i = 1, 2, \dots, n$

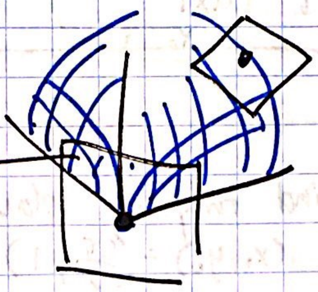
(ii) $$\lim_{\vec{x}\to\vec{x}_0} \frac{\| f(\vec{x}) - f(\vec{x}_0) - Df(\vec{x}_0)(\vec{x}-\vec{x}_0)\|}{\|\vec{x}-\vec{x}_0\|} = 0.$$

$$Df(\vec{x}_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \ddots & & \vdots \\ \vdots & & \ddots & \\ \frac{\partial f_m}{\partial x_1} & \cdots & & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

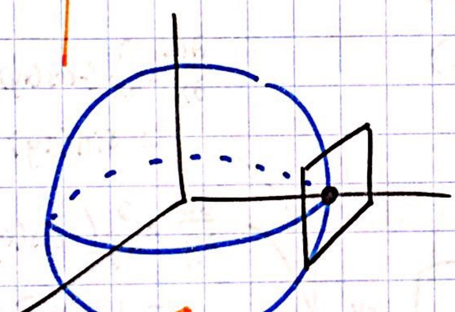


$f(x) = x^{1/3}$. $\left.\dfrac{df}{dx}\right|_{x=0} \overset{(\infty)}{=}$ DNE.

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} =$$

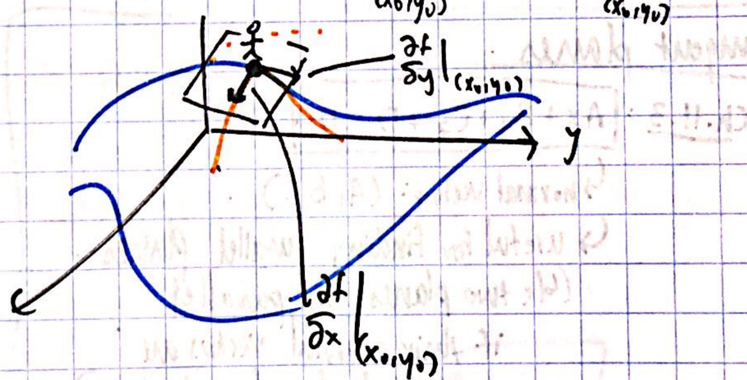


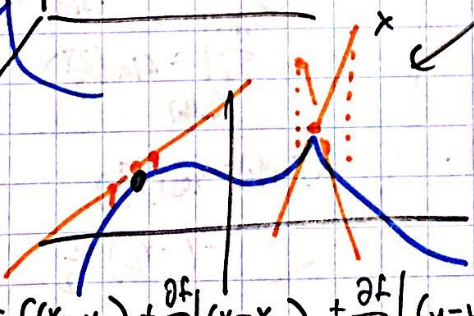
$f: \mathbb{R}^2 \to \mathbb{R}$ — graph looks like a surface in $\mathbb{R}^3$



tangent plane @ $(x_0, y_0, f(x_0,y_0))$.

$$z = f(x_0,y_0) + \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}(x-x_0) + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}(y-y_0)$$



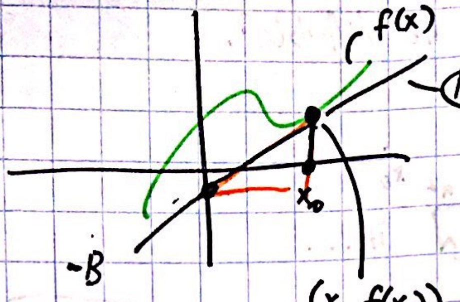


$z = ax + by + c$

$$z = f(x_0,y_0) + \underbrace{\left.\frac{\partial f}{\partial x}\right|_{(\dots)}}_{M}(x-x_0) + \underbrace{\left.\frac{\partial f}{\partial y}\right|_{(\dots)}}_{N}(y-y_0)$$

(M, N are real #'s)

$$= f(x_0,y_0) + M(x-x_0) + N(y-y_0) = \left(f(x_0,y_0) - Mx_0 - Ny_0\right) + Mx + Ny$$

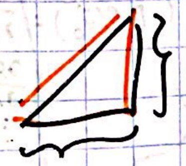


Linear approximation.

lines...

$y = Mx + B$.

$\frac{df}{dx}\Big|_{(x=x_0)}$ ... ? ... $f(x_0) - Mx_0$.



$M = \text{slope} = \frac{f(x_0) - B}{x_0}$.

$\Rightarrow Mx_0 = f(x_0) - B$.

$\Rightarrow B = f(x_0) - M(x_0)$

$= Mx + f(x_0) - Mx_0$

$\boxed{y = M(x - x_0) + f(x_0)}$.

lin. approx. of a 1-dim. function.

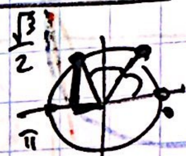
$\boxed{y = \frac{df}{dx}\Big|_{x_0}(x - x_0) + f(x_0)}$

plane

$z = \underbrace{ax + by}_{} + c$

$z = \frac{\partial f}{\partial x}\Big|_{(x_0, y_0)}(x - x_0) + \frac{\partial f}{\partial y}\Big|_{(\ldots)}(y - y_0) + f(x_0, y_0)$

$G(x, y) = -\cos(xy)$ .... Find tangent plane @ $(x_0, y_0) = \left(\frac{2\pi}{3}, 1\right)$



$z = \frac{\partial G}{\partial x}\Big|_{(\ldots)}(x - x_0) + \frac{\partial G}{\partial y}\Big|_{(\ldots)}(y - y_0) + G(x_0, y_0)$

$= \left(\frac{\sqrt{3}}{2}\right)\left(x - \frac{2\pi}{3}\right) + \left(\frac{\pi\sqrt{3}}{3}\right)(y - 1) + \left(\frac{1}{2}\right)$

$\frac{\partial G}{\partial x} = \frac{\partial}{\partial x}(-\cos(xy))$

$= \sin(xy) \cdot y$

$\frac{\partial G}{\partial y} = \frac{\partial}{\partial y}(-\cos(xy))$

$= \sin(xy) \cdot x$

ev.

$\frac{\partial G}{\partial x}\Big|_{(x_0, y_0)} = \sin\left(\frac{2\pi}{3} \cdot 1\right) \cdot 1$

$= \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$\frac{\partial G}{\partial y}\Big|_{(x_0, y_0)} = \sin\left(\frac{2\pi}{3}\right) \cdot \frac{2\pi}{3} = \frac{\sqrt{3}}{2} \cdot \frac{2\pi}{3}$

$= \frac{\pi\sqrt{3}}{3}$.

$G(x_0, y_0) = -\cos\left(\frac{2\pi}{3}\right)$

$= -\left(-\frac{1}{2}\right) = \frac{1}{2}$

---

tangent planes.

Ch. 11.3: $\boxed{Ax + By + Cz + D = 0}$

- normal vector: (A, B, C).
- useful for finding parallel planes.. (b/c two planes are parallel if their normal vectors are multiples of each other).

$z = ax + by + c$ — nice when finding tangent planes. (b/c we have a nice formula)

13.3 HW Q15) $z = xy^3 + 8y^{-1}$ is the graph of a function.

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$

$(x,y) \longmapsto xy^3 + 8y^{-1}$.

$\text{graph}(f) = \{(x,y,z) : z = f(x,y)\} \subset \mathbb{R}^3$

$= \{(x,y,xy^3+8y^{-1})\}$.

Want where tangent planes of this guy are parallel to the plane

$$P = \{(x,y,z) : 3x+5y+3z=0\} \subset \mathbb{R}^3.$$

To find parallel planes... we want planes w/ a multiple of the normal vector to P. $(3,5,3)$

We want tangent planes of our graph... say @ (a,b)

$$z = \frac{\partial f}{\partial x}\Big|_{(a,b)}(x-a) + \frac{\partial f}{\partial y}\Big|_{(a,b)}(y-b) + f(a,b)$$

$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(xy^3 + 8y^{-1}) = y^3$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(xy^3 + 8y^{-1}) = 3xy^2 - 8y^{-2}$

$$\boxed{z = b^3(x-a) + (3ab^2 - 8b^{-2})(y-b) + (ab^3 + 8b^{-1})}$$

Want the normal vector for this guy

(let's turn it into the form... $Ax + By + Cz + D = 0$.

$$b^3(x-a) + (3ab^2 - 8b^{-2})(y-b) - z + (ab^3 + 8b^{-1}) = 0$$

$$b^3x + (3ab^2-8b^{-2})y + (-1)z + \underbrace{(ab^3 + 8b^{-1} - ab^3 + (-b)(3ab^2 - 8b^{-2}))} = 0.$$

$\Rightarrow$ normal vector is $\langle b^3, (3ab^2-8b^{-2}), -1\rangle$.

We want this to be a multiple of $\langle 3,5,3\rangle$.

ie. we want $\langle 3,5,3\rangle = C\langle b^3, (3ab^2 - 8b^{-2}), -1\rangle$.

constant

$= \langle Cb^3, C(3ab^2 - 8b^{-2}), -C\rangle$. $\Rightarrow 3 = -C$ $\Rightarrow C = -3$

$= \langle -3b^3, -3(3ab^2 - 8b^{-2}), 3\rangle$

Compare components to solve for a & b.

$3 = -3b^3 \rightsquigarrow b^3 = -1$, ie. $\boxed{b = -1}$

$5 = -3(3ab^2 - 8b^{-2})$

$5 = -3(3a(-1)^2 - 8(-1)^{-2})$

$= -3(3a - 8)$

$= -9a + 24$

$\Rightarrow 9a = 19$

$\boxed{a = 19/9.}$

So our point $(a,b,f(a,b))$

$\left(\frac{19}{9}, -1, f\left(\frac{19}{9}, -1\right)\right)$

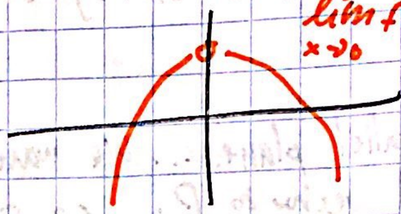
$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{(x+y)^2-(x-y)^2} \qquad 0 \le \frac{x^2y^2}{(x+y)^2-(x-y)^2} \le$$

$$(x+y)^2-(x-y)^2 = \cancel{x^2}+2xy+\cancel{y^2}-(\cancel{x^2}-2xy+\cancel{y^2}) = 4xy.$$

$$\frac{x^2y^2}{4xy} = \frac{xy}{4}.$$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{4} = 0.$$

$\lim_{x\to x_0} f =$

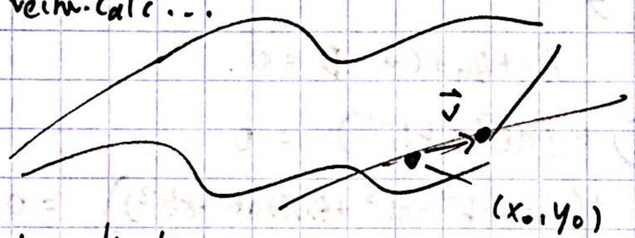


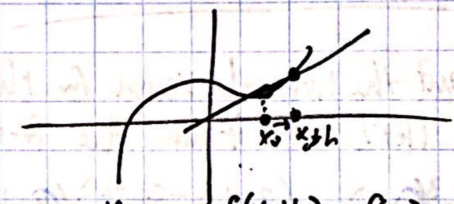
13.3) (theorem 9): $\left(f: U \to \mathbb{R}^m,\ U \subset \mathbb{R}^n \right)$

is <u>differentiable</u> if all the partials $\frac{\partial f_i}{\partial x_j}$ exist & are continuous @ $\vec{x}_0$.

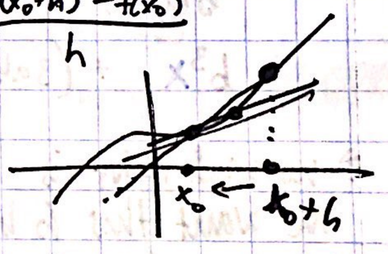
$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots \\ \vdots & \ddots \end{bmatrix}.$$

In vector calc...





$$\left.\frac{df}{dx}\right|_{x_0} = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$



Pick a direction ...

$$\left.\text{"}\frac{\partial f}{\partial \vec{v}}\text{"}\right|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f\left((x_0,y_0)+h\vec{v}\right) - f(x_0,y_0)}{h}$$

"directional derivative".

ε-δ.

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