∞-Cosmoi

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X^{\sim} \\ \pi \arrow[ur, dashed] & \end{tikzcd}$$

$$\begin{tikzcd} \Lambda^i[n] \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

∞-Cosmoi. (But first, some quasi-categorical background)

Def A simplicial set X is a quasi-category when given any map $\Lambda^i[n] \to X$ for $0<i<n$, there exists some (red) lift

$$\begin{tikzcd} \Lambda^i[n] \arrow[r] \arrow[d, hook] & X \\ \Delta[n] \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

In a quasi-category, we may think of 0-simplices (vertices) as objects, 1-simplices

$$\overset{a}{\bullet} \longrightarrow \overset{b}{\bullet}$$

as morphisms from a to b, and horn filling gives "composites" which are not unique, but are "up to homotopy", for given any

$$\Lambda^1[2] \longrightarrow X,$$

$$\begin{tikzcd} & b \arrow[dr, "g"] & \\ a \arrow[ur, "f"] \arrow[rr, dashed, "h"'] & & c \end{tikzcd}$$

$\exists$ an h filling the horn, which we view as a composite of g and f.

"homotopy" as Amartya explained looks like a 2-simplex

$$\begin{tikzcd} & b \arrow[dr, equal] & \\ a \arrow[ur, "f"] \arrow[rr, "g"'] & & b \end{tikzcd}$$

(= is a degenerate 1-simplex).

In particular, if X is a category, the nerve NX is a quasi-category with unique horn filling (higher horns witness associativity).

We have several special cases.

Note that $\underbrace{\Delta[n]}_{?} = N[n+1]$, where $[n+1]$ (bold font) is the category $1 \to 2 \to \dots \to n+1$, $[0 < \dots < n]$

in particular, $\mathbb{1} = \Delta[0]$ is a point, and a morphism $\mathbb{2} = \Delta[1] \longrightarrow X$ picks out a morphism in X

(we identify a category with its nerve and don't write NC anymore).

We write $\mathbb{I}$ to mean the "free-living isomorphism", that is, the nerve of the category

$$\begin{tikzcd} \bullet \arrow[r, bend left, "f"] & \bullet \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

$$\begin{tikzcd} {} \arrow[r, "f"] & \bullet \arrow[r, "f^{-1}"] & \cdots \arrow[r, "f"] & \bullet \\ \bullet \arrow[r, "f"] & \bullet \arrow[r, "f"] & \cdots \arrow[r, "f"] & \bullet \end{tikzcd}$$

<u>Definition</u> A map $f: X \to Y$ of simplicial sets is an <u>isofibration</u> if whenever we have the following commutative squares, there exists some dotted arrows making the diagrams commute

$$\begin{tikzcd} \Lambda^i[n] \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \Delta[n] \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd} \quad (0<i<n)$$

$$\begin{tikzcd}
\mathbb{1} = \Delta[0] \arrow[r] \arrow[d, hook] & X \arrow[d, two heads, "f"] \\
\mathbb{I} \arrow[r] \arrow[ur, dashed] & Y.
\end{tikzcd}$$

$\bullet \xrightarrow{\sim} \bullet$

$\bullet \overset{\sim}{\twoheadrightarrow} \bullet$

We denote isofibrations by $\twoheadrightarrow$, two headed arrows

Note that a simplicial set X is a quasi-category if and only if the unique map is an isofibration.

$$\begin{tikzcd}
\mathbb{1} \arrow[r] \arrow[d, hook] & X \arrow[d, two heads] \\
\mathbb{I} \arrow[r] \arrow[ur, dashed] & \mathbb{1}
\end{tikzcd}$$

[redacted] #1 black box number 1:

If $X \overset{f}{\hookrightarrow} Y$ is a map of simplicial sets, and $A \xrightarrow{p} B$ a map of quasi-categories, then if p is an isofibration and f is a monomorphism, the map $f \widehat{\pitchfork} p$ is an isofibration

$$\begin{tikzcd}
A^Y \arrow[dr, dashed, two heads, "f \widehat{\pitchfork} p"'] \arrow[drr, bend left, two heads] \arrow[ddr, bend right, two heads] & & \\
& A^X \times_{B^X} B^Y \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & A^X \arrow[d, two heads, "p_*"] \\
& B^Y \arrow[r, two heads, "f^*"'] & B^X .
\end{tikzcd}$$

The other maps marked with $\twoheadrightarrow$ can be seen to be isofibrations as special cases of the above. For example, to see $A^Y \to A^X$ is an isofibration take $B = \mathbb{1}$.

$$\begin{tikzcd}
A^Y \arrow[r, two heads] \arrow[rr, bend left] \arrow[dr, bend right] & A^X \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & A^X \arrow[d, two heads] \\
& \mathbb{1} \arrow[r] & \mathbb{1}
\end{tikzcd}$$

and taking $X = \emptyset \hookrightarrow Y$ shows A^Y is a quasi-category.

$$A^Y \twoheadrightarrow A^{\emptyset} = \mathbb{1}$$

Definition A functor (= map of sSets) $f : A \to B$ between quasi-categories is an equivalence if there exists a map $g : B \to A$, and maps α, β making the following diagrams commute

$$
\begin{tikzcd}
 & A \\
A \arrow[ur, equal] \arrow[r, "\alpha"] \arrow[dr, "gf"'] & A^{\mathbb{I}} \arrow[u, "ev_0"'] \arrow[d] \\
 & A
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & B \\
B \arrow[ur, "fg"] \arrow[r, "\beta"] \arrow[dr, equal] & B^{\mathbb{I}} \arrow[u] \arrow[d, "ev_1"] \\
 & B
\end{tikzcd}
$$

i.e., this is the data of being a quasi-inverse for f and natural isomorphisms witnessing the inverse equivalence.

We denote equivalences by $\xrightarrow{\sim}$.

Def A map $f: A \to B$ of simplicial sets is a trivial fibration if given a commuting square, there is a dashed diagonal map making the diagram commute:

$$\begin{tikzcd} \partial\Delta[1] \arrow[r] \arrow[d, hook] & A \arrow[d, "f"] \\ \Delta[1] \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

.

We denote trivial fibrations by $\overset{\sim}{\twoheadrightarrow}$.

(Stacks Project): A map $A \xrightarrow{f} B$ of sSets is a trivial fibration if and only if whenever $X \hookrightarrow Y$ is a monomorphism together with a commuting solid square, there is a diagonal lift.

$$\begin{tikzcd} X \arrow[r] \arrow[d, hook] & A \arrow[d] \\ Y \arrow[r] \arrow[ur, dashed, "\exists"] & B \end{tikzcd}$$

Prop For a map of quasi-categories $f: A \to B$, TFAE

(i) f is a trivial fibration

(ii) f is an isofibration and an equivalence

(iii) f is an isofibration and "split fiber homotopy equivalence", i.e., there is a $g: B \to A$ with $fg = id_B$, and a natural isomorphism from gf to id_A composing with f to the identity, i.e.,

$$
\begin{tikzcd}
A + A \arrow[rr, "{(gf, \mathrm{id}_A)}"] \arrow[d] & & A \arrow[d, "f"] \\
A \times \mathbb{I} \arrow[r, "\pi"'] \arrow[urr, dashed, "\alpha"] & A \arrow[r, "f"'] & B
\end{tikzcd}
$$

π = constant homotopy

Proof. (i) $\Rightarrow$ (ii) + (iii) It is clear that f is an isofibration when it is a trivial fibration, So we check (iii), which clearly implies (ii).

We use the (left) monomorphism

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d, hook] & A \arrow[d, "s"', "f"] \\
B \arrow[r, equal] \arrow[ur, dotted, "g"] & B
\end{tikzcd}$$

$fg = id_B$

$$\begin{array}{c} \Lambda^i[1] \\ \downarrow \\ \Delta[1] \end{array}$$

$$\begin{array}{c} \mathbb{I} \\ \downarrow \\ \mathbb{I} \end{array}$$

to get a lift g such that $fg = 1_B$.

Now we can form the diagram

$$\begin{tikzcd}
A+A \arrow[rr, "{(gf, \mathrm{id}_A)}"] \arrow[d, hook] & & A \arrow[d, "f", "\sim"'] \\
A \times \mathbb{I} \arrow[r, "\pi"'] \arrow[urr, dashed] & A \arrow[r, "f"'] & B
\end{tikzcd}$$

and again use lifting against monomorphisms to prove the claim.

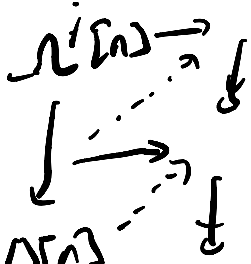
(iii) $\Longrightarrow$ (ii) is clear

(ii) $\Longrightarrow$ (i) is hard

blackbox

■ ← currently a □

Proposition ([redacted]) If $f: A \to B$ is an isofibration of quasi-categories, and if $i: X \hookrightarrow Y$ is a monomorphism of simplicial sets, then if either f is a trivial fibration, or i is in the class cellularly generated by the inner horn inclusions $\Lambda^i[n] \hookrightarrow \Delta[n]$ $0<i<n$ and the inclusion $\mathbb{1} \hookrightarrow \mathbb{I}$, then the map $i \widehat{\pitchfork} f$ is a trivial fibration



$$\begin{tikzcd}
A^Y \arrow[r, two heads, "\sim", "i \widehat{\pitchfork} f"'] \arrow[rr, bend left] \arrow[dr, bend right, two heads] & B^Y \times_{B^X} A^X \arrow[r, two heads] \arrow[d, two heads] & A^X \arrow[d, two heads, "f_*"] \\
& B^Y \arrow[r, two heads, "i^*"'] & B^X
\end{tikzcd}$$

$$\begin{tikzcd}
A^{\Lambda^i[n]} \\
A^{\Delta[n]} \arrow[u, "\sim"]
\end{tikzcd}$$

Finally,

<u>Definition</u> (∞-cosmoi) An ∞-cosmos $\mathcal{K}$ is a quasi-categorically enriched category, i.e., a simplicially enriched category whose hom-sets are all quasi-categories, denoted $\mathrm{Fun}(A, B)$, such that $\mathcal{K}$ has a distinguished class of morphisms called isofibrations and denoted $\twoheadrightarrow$, which are closed under composition and contain isomorphisms, such

that

- $\mathcal{K}$ has a terminal object, all small products, pullbacks along isofibrations, inverse limits of countable towers of isofibrations, and has simplicial cotensors, i.e., to every simplicial set X, and $A, B \in \mathcal{K}$, there is an object $B^X \in \mathcal{K}$ such that

$$sSet(X, Fun(A,B)) \cong Fun(A, B^X),$$

as simplicial sets.

- Isofibrations are required to be closed under pullbacks, products, inverse limits of towers, and Leibniz cotensors with monomorphisms of simplicial sets. Additionally, we ask that if $f : A \twoheadrightarrow B$ is an isofibration in $\mathcal{K}$, and X is any object of $\mathcal{K}$, the map $\mathrm{Fun}(X, A) \twoheadrightarrow \mathrm{Fun}(X, B)$ is an isofibration of quasi-categories.

$$X \twoheadrightarrow \mathbb{1}.$$

Now, we should unpack these definitions a little.

$\mathcal{X}$ is simplicially enriched.

For all $A, B \in \mathcal{X}$, $\mathrm{Fun}(A,B)_\bullet$, which is in sSet, such that we have distinguished morphisms

$$\mathbb{1} \xrightarrow{1_A} \mathrm{Fun}(A,A),$$

$$\mathrm{Fun}(B,C) \times \mathrm{Fun}(A,B) \longrightarrow \mathrm{Fun}(A,C)$$

$$\begin{tikzcd}
\mathrm{Fun}(C,D) \times (\mathrm{Fun}(B,C) \times \mathrm{Fun}(A,B)) \arrow[d] \\
(\mathrm{Fun}(C,D) \times \mathrm{Fun}(B,C)) \times \mathrm{Fun}(A,B)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Fun}(A,B) \arrow[r, phantom, "\cong"] \arrow[rrr, bend right, equal] & \mathbb{1} \times \mathrm{Fun}(A,B) \arrow[r, "\mathrm{id}_B"] & \mathrm{Fun}(B,B) \times \mathrm{Fun}(A,B) \arrow[r, "\circ"] & \mathrm{Fun}(A,B)
\end{tikzcd}$$

$n=0$, $\mathrm{Fun}(A,B)_0$, gives an ordinary category K_0. The "underlying category of K). $\mathrm{Fun}(A,B)_n$ as a hom-set from A to B, this is going to give us a category K_n.

$$\cdots K_2 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; K_1 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; K_0$$

$$\widehat{\lim}\, \mathrm{Fun}(A, B_n) \longrightarrow \mathrm{Fun}(A, \widehat{\lim}\, B),$$

Simplicial cotensor. If $A \in \mathcal{K}$, X is a simplicial set, we can form the simplicial cotensor A^X, which we want to satisfy

$$\mathrm{sSet}(X, \mathrm{Fun}(B, A)) \cong \mathrm{Fun}(B, A^X)$$

we want this to be functorial.

$$\mathrm{Set}(X, \mathrm{Set}(Y, Z)) \cong \mathrm{Set}(X \times Y, Z)$$
$$\cong \mathrm{Set}(Y, Z^X) \qquad Z^X = \mathrm{Set}(X, Z).$$

$$\underline{s\mathrm{Set}}(X, Y)_1 = s\mathrm{Set}(X \times \Delta[1], Y).$$

Definition We say that a map $f: A \to B$ in an ∞-cosmos $\mathcal{K}$ is an equivalence if $\mathrm{Fun}(X, A) \to \mathrm{Fun}(X, B)$ is an equivalence of quasi-categories, for $X \in \mathcal{K}$.

$\mathrm{Fun}(A,B)$. Generally we call $A \in \mathcal{K}$ in an ∞-cosmos $\mathcal{K}$ ∞-categories, and maps $f: A \to B$ of ∞-categories are called ∞-functors, or just functors.

<u>Examples</u>

<u>Claim</u> The category $\mathcal{Q}Cat$ of quasi-categories forms an ∞-cosmos.

Proof. $\mathrm{Fun}(A,B) =: B^A$ $(sSet(A,B))$, for any simplicial set A, B^A is a quasi-category, so in particular, this gives quasi-categorical enrichment and simplicial cotensors.

$$
\begin{aligned}
sSet(X, \mathrm{Fun}(A,B)) &= sSet(X, sSet(A,B)) \\
&\cong sSet(X \times A, B) \\
&\cong sSet(A, sSet(X,B)) \\
&\cong \mathrm{Fun}(A, B^X).
\end{aligned}
$$

$$
\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[rr, bend left] & B \wedge C \arrow[r] \arrow[d] & B \arrow[d] \\
Y \arrow[r] \arrow[ur, dashed] \arrow[urr, dashed] \arrow[rr, bend right] & C \arrow[r] \arrow[d] & A \\
& \underline{1} &
\end{tikzcd}
$$

$\Lambda^i[n] \hookrightarrow \Delta[n]$

$\underline{1} \hookrightarrow \mathbb{I}$

$\lim \leftarrow \left(\twoheadrightarrow \quad \twoheadrightarrow \quad \twoheadrightarrow \quad \twoheadrightarrow \right)$

$\downarrow$

$A_n \twoheadrightarrow \underline{1}$

cotensors, products ✓

$$\begin{tikzcd} \Lambda^i[n] \arrow[r] \arrow[d] & \prod_J A_j \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd}$$

$$\begin{tikzcd} \Lambda^i[n] \arrow[r] \arrow[d] & A_j \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd}$$

$\square$

__Claim__ Cat, the category of 1-categories, forms an ∞-cosmos

Proof. $\prod A_j$,

$$
\begin{tikzcd}
\partial\Delta[n] \arrow[r] \arrow[d, hook] & P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\
\Delta[n] \arrow[r] \arrow[ur, dashed] & C \arrow[r] & B \arrow[d] \\
& & 1
\end{tikzcd}
$$

$$A^X \qquad A^X = sSet(X, NA)_n$$

$$\cup\!\!\!\!\to$$

$$\cong Cat(hX, A) \cong N(Cat(hX, A)). \quad \square$$

Kan the category of Kan complex

$$\begin{tikzcd} \Lambda^i[n] \arrow[r] \arrow[d, hook] & A \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd} \qquad \forall\, 0 \le i \le n.$$

$$Kan \hookrightarrow QCat.$$

Homotopy 2-category

Given an ∞-cosmos $\mathcal{K}$, we can form the homotopy 2-category $h\mathcal{K}$, which has as objects, the same objects as in $\mathcal{K}$, and as morphisms, $\underbrace{hFun(A,B)} := h(Fun(A,B))$.

$$\begin{tikzcd} 2\text{-Cat} \arrow[r, bend right, "N"'] \arrow[r, phantom, "\perp"] & \text{sSet-Cat} \arrow[l, bend right, "h"'] \end{tikzcd}$$

$\bullet \overset{\simeq}{\dashrightarrow}$

$A \simeq B$ in $h\mathcal{K}$

$\Rightarrow$ equivalence in $\mathcal{K}$

Proposition For a functor $f: A \to B$ in an ∞-cosmos $\mathcal{K}$, TFAE:

(i) f is an equivalence in $\mathcal{K}$, i.e., $\mathrm{Fun}(X, A) \to \mathrm{Fun}(X, B)$ is an equivalence of quasi-categories for all $X \in \mathcal{K}$.

(ii) f is an equivalence in $h\mathcal{K}$, i.e., there is $g: B \to A$, and invertible 2-cells (natural isomorphisms) $\alpha: \mathrm{id}_A \Rightarrow gf$, $\beta: fg \Rightarrow \mathrm{id}_B$.

(iii) f is an equivalence internal to $\mathcal{K}$, i.e., there is a functor $g : B \to A$ in $\mathcal{K}$, and functors making making the diagrams

$$\begin{tikzcd}
 & A \\
A \arrow[ur, equal] \arrow[r, "\alpha"] \arrow[dr, "gf"'] & A^{\mathbb{I}} \arrow[u, "ev_0"'] \arrow[d, "ev_1"] \\
 & A
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & B \\
B \arrow[ur, "fg"] \arrow[r, "\beta"] \arrow[dr, equal] & B^{\mathbb{I}} \arrow[u, "ev_0"'] \arrow[d, "ev_1"] \\
 & B
\end{tikzcd}$$

Proof. (i)$\Rightarrow$(ii) Suppose f is an equivalence in $\mathcal{K}$.

<u>Claim</u> If $\varphi: X \to Z$ is an equivalence of quasi-categories, then $\varphi: hX \to hZ$ is an equivalence of categories

Proof of Claim. $\psi: Z \to X$,

$$\begin{tikzcd} & X \\ X \arrow[r, "\gamma"] \arrow[ur, equal] \arrow[dr, "\psi\varphi"'] & X^{\mathbb{I}} \arrow[u, "ev_0"'] \arrow[d, "ev_1"] \\ & X \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} & & h(X) \\ hX \arrow[r, "\gamma"] \arrow[urr, equal] \arrow[drr, "\psi\varphi"'] & h(X^{\mathbb{I}}) \arrow[r] & h(X)^{\mathbb{I}} \arrow[u, "ev_0"'] \arrow[d, "ev_1"] \\ & & h(X). \end{tikzcd}$$

$\square$

So, we have $\mathrm{Fun}(X, A) \to \mathrm{Fun}(X, B)$ is an equivalence of quasi-categories for all $X \in \mathcal{K}$. So upon passing to homotopy categories, $h\mathrm{Fun}(X, A) \to h\mathrm{Fun}(X, B)$ is an equivalence of categories.

$h\mathrm{Fun}(B, A) \xrightarrow{f_*} h\mathrm{Fun}(B, B)$ is an equivalence, $\mathrm{id}_B \in$, $g \in h\mathrm{Fun}(B, A)$

$fg \simeq \mathrm{id}_B$,

$$
\begin{tikzcd}
\mathrm{hFun}(A,A) \arrow[rr, "f_*"] & & \mathrm{hFun}(A,B) \\
\overset{\psi}{gf} \arrow[rr, mapsto] \arrow[d, phantom, "\cong"] & & fgf \arrow[d, phantom, "\cong"] \\
\mathrm{id}_A \arrow[rr, mapsto] \arrow[urr, phantom, "? \;\Leftarrow"] & & f.
\end{tikzcd}
$$

(ii) $\Rightarrow$ (iii) $f, g : B \to A$, $\alpha : \mathbb{I} \to \mathrm{hFun}(A,A)$

$\mathrm{id}_A \Rightarrow gf$.

Fact An arrow $\varphi : x \to y$ in a quasi-category is an isomorphism iff

$$
\begin{tikzcd}
2 \arrow[r] \arrow[d, hook] & A \\
\mathbb{I} \arrow[ur, dashed] &
\end{tikzcd}
$$

$\square$

$$
\begin{tikzcd}
 & & 2 \arrow[r] \arrow[d, hook] & \mathrm{Fun}(A, A) \\
2 \arrow[urr, phantom] \arrow[rr] & & \mathbb{I} \arrow[ur, dashed] & \\
 & \mathbb{I} \arrow[r] & \mathbb{I} + \underline{1} \arrow[u, hook] \arrow[uur, "{(\mathrm{id}_A, gf)}"'] &
\end{tikzcd}
$$

$$\begin{tikzcd} & A^{I} \cong A \\ A \arrow[ur, equal] \arrow[r, "\alpha"] \arrow[dr, bend right, "gf"'] & A^{I} \arrow[u, "\pi\, ev_0"'] \arrow[d, "ev_1"] \\ & A . \end{tikzcd}$$

(iii) $\Rightarrow$ (i), $f : A \to B$, $g : B \to A$

$$\mathrm{Fun}(X, A) \xrightarrow{f_*} \mathrm{Fun}(X, B) \xrightarrow{g_*} \mathrm{Fun}(X, A)$$

$$\mathrm{Fun}(X, A^{\mathbb{I}}) \simeq \mathrm{sSet}(\mathbb{I}, \mathrm{Fun}(X,A)) \simeq$$

$$\mathrm{Fun}(X,A)^{\mathbb{I}}$$

$$\begin{tikzcd}
 & A \\
A \arrow[ur, equal] \arrow[r] & A^{\mathbb{I}} \arrow[u, "ev_0"'] \arrow[d, "ev_1"] \\
 & A
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
 & \mathrm{Fun}(X,A) \\
\mathrm{Fun}(X,A) \arrow[ur, equal] \arrow[r] \arrow[dr, "\exists f_*"'] & \mathrm{Fun}(X,A^{\mathbb{I}}) \arrow[u, "ev_0"'] \arrow[d, "ev_1"] \\
 & \mathrm{Fun}(X,A).
\end{tikzcd}
\quad \square$$

$$f : \underbrace{A \times B\{3\}}\to B$$