Joyal's Lifting Theorem

∞-Categories Reading Group, Cisinski Reading Seminar

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Some Definitions

Definition (Kan Complex)

X is **Kan complex** if all horns have fillers, that is $X \to \Delta^0$ is a Kan fibration.

Definition (Quasicategory)

X is **quasicategory** if all inner horns have fillers, that is $X \to \Delta^0$ is an inner fibration.

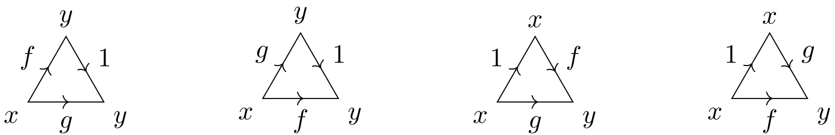
Definition (Right/Left Fibration)

$f: X \to Y$ is **right/left fibration** if all right/left horns have fillers.

From earlier Meetings

Definition (Homotopy Category)

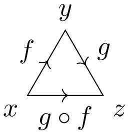
Let X be a quasicategory. Define an equivalence relation called the **homotopy relation** on $X_1(x, y)$ by $f \sim g$ whenever any (and then all) of the triangles

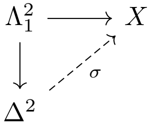


exist in X_2. Define a category τX by

- objects = the 0-simplices, morphisms = $\tau X(x, y) := X_1(x, y)/\sim$,
- composition via

$$\tau X(x, y) \times \tau X(y, z) \to \tau X(x, z)$$
$$([f], [g]) \to [d_1(\sigma)]$$





Little Lemma

Definition (Isomorphism)

Edge $f\colon x \to y$ in quasicategory X is called **isomorphism** if its corresponding arrow in τX g is one.

Lemma

If $p\colon X \to Y$ is a right fibration between quasicategories, then an isomorphism $\varphi\colon p(x) \to y$ in Y, having domain in the image of p, lifts to X.

Proof.

A lift in the diagram below yields a candidate for an inverse of a lift of φ to X.

$$\begin{array}{ccc} \{1\} & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \\ \Delta^1 & \xrightarrow{\varphi^{-1}} & Y \end{array}$$

Solving some more appropriate lifting problems gives then an isomorphism that lifts φ. □

Join – Categorically

The monoidal structure on the augmented simplex category

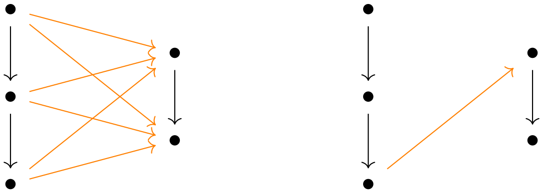
$$\mathbf{\Delta}_+ = \text{finite ordinals} \subseteq \mathbf{Cat},$$

given by $[m] + [n] = [m+1+n]$ with unit object $[-1]$ induces the join of augmented simplicial sets as Day convolution, that is the left Kan extension in

$$\begin{array}{ccc} \mathbf{\Delta}_+ \times \mathbf{\Delta}_+ & \xrightarrow{\;+\;} & \mathbf{\Delta}_+ \\ \downarrow & & \downarrow \\ \mathbf{sSet}_+ \times \mathbf{sSet}_+ & \overset{\star}{\dashrightarrow} & \mathbf{sSet}_+, \end{array}$$

where the vertical functors are the Yoneda embeddings. One can define the join of ordinary simplicial sets by first taking the terminal augmentation and then the join as above.

Join – Geometrically



$$\Delta^i \star \Delta^j = \Delta^{i+1+j}$$

$$(X \star Y)_n = \coprod_{i+1+j=n} X_i \times Y_j \text{ with } X_{-1} = \{*\} = Y_{-1}$$

Face maps are

$$d_k \colon X_i \times Y_j \ni (\sigma, \tau) \mapsto \begin{cases} (d_k(\sigma), \tau), & 0 \leq k \leq i, \\ (\sigma, d_{k-(n-j)}(\tau)), & n-j \leq k \leq n \end{cases}$$

and similar for degeneracy maps.

Combinatorial Lemma

Lemma

For $0 \leq k \leq n$, a pushout in the square

$$\begin{array}{ccc} \Lambda^n_k \star \partial\Delta^m & \longrightarrow & \Lambda^n_k \star \Delta^m \\ \downarrow & & \downarrow \\ \Delta^n \star \partial\Delta^m & \longrightarrow & \Delta^n \star \Delta^m \end{array}$$

yields

$$\begin{aligned} \Lambda^{n+1+m}_k &\simeq \Lambda^n_k \star \Delta^m \cup_{\Lambda^n_k \star \partial\Delta^m} \Delta^n \star \partial\Delta^m \\ &\subseteq \Delta^n \star \Delta^m \\ &\simeq \Delta^{n+1+m}. \end{aligned}$$

Slice

Joining with a fixed T does not commute with colimits but

$$\begin{aligned} \mathbf{sSet} &\to T/\mathbf{sSet} \\ X &\mapsto (T \to X \star T) \end{aligned}$$

does. Thus we wish ourselves a right adjoint, called **under slice**

$$\begin{aligned} T/\mathbf{sSet} &\to \mathbf{sSet} \\ (T \to X) &\mapsto X/T \end{aligned}$$

and calculate

$$\begin{aligned} (X/T)_n &\simeq \mathbf{sSet}(\Delta^n, X/T) \\ &\simeq T/\mathbf{sSet}(T \to \Delta^n \star T, T \to X). \end{aligned}$$

Take this as definition and do the same with $T \to T \star \_$ to obtain also $T\backslash\_$, called **over slice**.

Key Technical Lemma

Lemma (Adjunction)

Given maps $i\colon K \to L$, $j\colon S \to T$, $p\colon X \to Y$ and in addition a map $T \to X$ there is a correspondence of lifting problems

$$\begin{array}{ccc} K \star T \cup_{K \star S} L \star S & \longrightarrow & X \\ {\scriptstyle i \hat{\star} j}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ L \star T & \longrightarrow & Y \end{array} \qquad \text{and} \qquad \begin{array}{ccc} K & \longrightarrow & X/T \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p/j} \\ L & \longrightarrow & X/S \times_{Y/S} Y/T, \end{array}$$

with the left diagram in $T/\mathbf{sSet}$ and the maps $i \hat{\star} j$ and p/j induced by

$$\begin{array}{ccc} K \star S & \longrightarrow & K \star T \\ \downarrow & & \downarrow \\ L \star S & \longrightarrow & L \star T \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X/T & \longrightarrow & X/S \\ \downarrow & & \downarrow \\ Y/T & \longrightarrow & Y/S. \end{array}$$

Dual of the Key Technical Lemma

Lemma (Adjunction)

Given maps $i\colon K \to L$, $j\colon S \to T$, $p\colon X \to Y$ and in addition a map $L \to X$ there is a correspondence of lifting problems

$$\begin{array}{ccc} K \star T \cup_{K \star S} L \star S & \longrightarrow & X \\ {\scriptstyle i \hat{\star} j}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ L \star T & \longrightarrow & Y \end{array} \quad \text{and} \quad \begin{array}{ccc} S & \longrightarrow & L\backslash X \\ {\scriptstyle j}\downarrow & \nearrow & \downarrow{\scriptstyle i\backslash p} \\ T & \longrightarrow & K\backslash X \times_{K\backslash Y} L\backslash Y, \end{array}$$

with the left diagram in $L/\mathbf{sSet}$ and the maps $i \hat{\star} j$ and $i\backslash p$ induced by

$$\begin{array}{ccc} K \star S & \longrightarrow & K \star T \\ \downarrow & & \downarrow \\ L \star S & \longrightarrow & L \star T \end{array} \quad \text{and} \quad \begin{array}{ccc} L\backslash X & \longrightarrow & K\backslash X \\ \downarrow & & \downarrow \\ L\backslash Y & \longrightarrow & K\backslash Y. \end{array}$$

Main Theorem

Theorem (Joyal's Lifting Theorem)

Let X be a quasicategory. Then there is a lift in

$$\begin{array}{ccc} \Lambda^n_0 & \longrightarrow & X \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

provided that

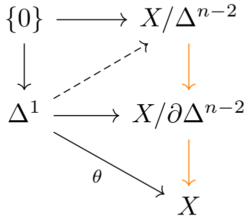
$$\begin{aligned} \theta \colon \Delta^1 &\simeq \Delta^1 \star \emptyset \\ &\to \Delta^1 \star \partial\Delta^{n-2} \\ &\to \{0\} \star \Delta^{n-2} \cup_{\{0\}\star\partial\Delta^{n-2}} \Delta^1 \star \partial\Delta^{n-2} \\ &\simeq \Lambda^n_0 && \text{(Combinatorial Lemma)} \\ &\to X \end{aligned}$$

is an isomorphism.

Main Proof

Proof.

By the Key Technical Lemma the lifting problem is equivalent to upper square in



Orange maps are right fibrations by the following slides. By the Little Lemma, θ lifts. □

Where does the last diagram come from?

$$\begin{array}{ccc} \{0\} & \longrightarrow & X/\Delta^{n-2} \\ \downarrow & \nearrow & \downarrow \\ \Delta^1 & \xrightarrow[\theta]{} & X/\partial\Delta^{n-2} \end{array} \qquad \text{is just} \qquad \begin{array}{ccc} K & \longrightarrow & X/T \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p/j} \\ L & \longrightarrow & X/S \times_{Y/S} Y/T, \end{array}$$

from the Key Technical Lemma for

$$(S \to T) = \left(\partial\Delta^{n-2} \to \Delta^{n-2}\right)$$

and $Y = \Delta^0$.

Where does the last diagram come from?

$$\begin{array}{ccc} \Delta^1 & \longrightarrow & X/\partial\Delta^{n-2} \\ & \overset{\theta}{\searrow} & \downarrow \\ & & X \end{array} \qquad \text{is just} \qquad \begin{array}{ccc} K & \longrightarrow & X/T \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p/j} \\ L & \longrightarrow & X/S \times_{Y/S} Y/T, \end{array}$$

from the Key Technical Lemma for

$$(S \to T) = \left(\emptyset \to \partial\Delta^{n-2}\right)$$

and $Y = \Delta^0$.

Restriction Map is Right Fibration 0/4

Proposition

If $p\colon X \to Y$ is an inner fibration and $j\colon S \to T$ is an inclusion, then p/j in

$$\begin{array}{ccc} K & \longrightarrow & X/T \\ {\scriptstyle i}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle p/j} \\ L & \longrightarrow & X/S \times_{Y/S} Y/T, \end{array}$$

is a right fibration.

Corollary

The map $X/T \to X/S$ and $X/S \to X$ in the Main Proof are right fibrations.

Restriction Map is Right Fibration 1/4

Let $0 < k \leq n$. Consider the chain of correspondences of lifting problems

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X/T \\ \downarrow & \nearrow & \downarrow {\scriptstyle p/j} \\ \Delta^n & \longrightarrow & X/S \times_{Y/S} Y/T \end{array} \quad \Longleftrightarrow \quad \begin{array}{ccc} \Lambda^n_k \star T \cup_{\Lambda^n_k \star S} \Delta^n \star S & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \Delta^n \star T & \longrightarrow & Y \end{array}$$

Restriction Map is Right Fibration 2/4

The Dual of the Key Technical Lemma gives

$$\begin{array}{ccc} \Lambda^n_k \star T \cup_{\Lambda^n_k \star S} \Delta^n \star S & \longrightarrow & X \\ \downarrow & \overset{\exists}{\nearrow} & \downarrow \\ \Delta^n \star T & \longrightarrow & Y \end{array} \iff \begin{array}{ccc} S & \longrightarrow & \Delta^n \backslash X \\ \downarrow & \overset{\exists}{\nearrow} & \downarrow \\ T & \longrightarrow & \Lambda^n_k \backslash X \times_{\Lambda^n_k \backslash Y} \Delta^n \backslash Y \end{array}$$

Restriction Map is Right Fibration 3/4

Since the class of monomorphisms is the saturation of the set of boundary inclusions $\partial\Delta^n \to \Delta^n$ one has

$$
\begin{array}{ccc}
S & \longrightarrow & \Delta^n\backslash X \\
\downarrow & \overset{\dashrightarrow}{} & \downarrow \\
T & \longrightarrow & \Lambda^n_k\backslash X \times_{\Lambda^n_k\backslash Y} \Delta^n\backslash Y
\end{array}
\quad \Longleftrightarrow \quad
\begin{array}{ccc}
\partial\Delta^m & \longrightarrow & \Delta^n\backslash X \\
\downarrow & \overset{\dashrightarrow}{} & \downarrow \\
\Delta^m & \longrightarrow & \Lambda^n_k\backslash X \times_{\Lambda^n_k\backslash Y} \Delta^n\backslash Y
\end{array}
$$

Restriction Map is Right Fibration 4/4

The Dual of the Key Technical Lemma gives

$$\begin{array}{ccc} \partial\Delta^m & \longrightarrow & \Delta^n\backslash X \\ \downarrow & \nearrow & \downarrow \\ \Delta^m & \longrightarrow & \Lambda^n_k\backslash X \times_{\Lambda^n_k\backslash Y} \Delta^n\backslash Y \end{array} \quad \Longleftrightarrow \quad \begin{array}{ccc} \Lambda^n_k \star \Delta^m \cup_{\Lambda^n_k \star \partial\Delta^m} \Delta^n \star \partial\Delta^m & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow p \\ \Delta^n \star \Delta^m & \longrightarrow & Y, \end{array}$$

By the Combinatorial Lemma pushout join map is $\Lambda^{n+1+m}_k \to \Delta^{n+1+m}$. The lift exists since p is an inner fibration and $0 < k \leq n$. $\square$

Main Corollary

Theorem

A quasicategory X is a Kan complex $\iff$ τX is a groupoid.

Proof.

One direction and the cases of inner horns for the other are trivial. The rest follows by Joyal's lifting theorem ($+$ explicitly checking the one-dimensional case). □

Core

Definition

For X a quasicategory let its **core** $X^{\simeq}$ be the simplicial subset consisting of those simplices with all their edges being isomorphisms. So there is a pullback

$$\begin{array}{ccc} X^{\simeq} & \longrightarrow & X \\ \downarrow & & \downarrow \\ N((\tau X)^{\simeq}) & \longrightarrow & N(\tau X). \end{array}$$

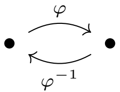
- $X^{\simeq}$ is the biggest Kan complex contained in X.

$$\begin{array}{ccc} K & \xrightarrow{\forall} & X \\ {\scriptstyle \exists!}\downarrow & \nearrow & \\ X^{\simeq} & & \end{array}$$

Characterization of Isomorphisms

Proposition

An edge in a quasicategory X is an isomorphism iff it comes from $\Delta^1 \hookrightarrow \mathrm{N}J \to X$, where J is the category



having exactly two non-degenerate simplices in each dimension:

- $0 \to 1 \to 0 \to \cdots \to 1/0$
- $1 \to 0 \to 1 \to \cdots \to 0/1$

(The geometric realization of this is the infinity sphere S^∞. As a remark, this gives a proof of S^∞ being contractible since J is equivalent to the terminal category and nerve and realization send natural transformations to homotopies.)

Characterization of Isomorphisms

Proof.

Let $\Delta^1 = J_1 \subseteq J_2 \subseteq \cdots \subseteq \bigcup_k J_k = \mathrm{N}J$, where J_k is simplicial subset generated by the k-simplex $\iota_k \colon 0 \to 1 \to 0 \to \cdots 0/1$.

There is a pushout

$$\begin{array}{ccc} \Lambda^n_0 & \longrightarrow & J_{k-1} \\ \downarrow & & \downarrow \\ \Delta^k & \xrightarrow{\iota_k} & J_k. \end{array}$$

Then if $\Delta^1 \to X$ is an isomorphism, it factors as in

$$\begin{array}{ccccc} \Delta^1 & \longrightarrow & X^{\simeq} & \longrightarrow & X \\ \downarrow & \nearrow & & & \\ \mathrm{N}J & & & & \end{array}$$

and the map lifts since $X^{\simeq}$ is Kan complex. □

Object-Wise Criterion for Natural Isomorphisms

Definition

Let X_0 be the constant simplicial set on the vertices of X. Then define $\mathrm{Fun}^{\simeq}(X,Y)$ via a pullback

$$\begin{array}{ccc} \mathrm{Fun}^{\simeq}(X,Y) & \longrightarrow & \mathrm{Fun}(X,Y) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(X_0,Y)^{\simeq} & \longrightarrow & \mathrm{Fun}(X_0,Y). \end{array}$$

Define $h(X,Y)$ via the (proposed existence of a) natural isomorphism

$$\mathbf{sSet}(X, h(Y,Z)) \cong \mathbf{sSet}(Y, \mathrm{Fun}^{\simeq}(X,Z)).$$

Proposition

For any quasicategories X, Y one has

$$\mathrm{Fun}^{\simeq}(X,Y) = \mathrm{Fun}(X,Y)^{\simeq}.$$

Important Lemma

Lemma (Cisinski, Thm. 3.5.8)

If Y is a quasicategory, then $\mathrm{eval}_1 \colon h(\Delta^1, Y) \to Y$ has the right lifting property against monomorphisms that induce a bijection on objects.

Proof.

It suffices to check that all lifting problems

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & h(\Delta^1, Y) \\ \big\downarrow & \nearrow & \big\downarrow \\ \Delta^n & \longrightarrow & Y \end{array}$$

admit a solution for $n > 0$. We get

$$\begin{array}{ccc} \Delta^1 \times \partial\Delta^n \cup \{1\} \times \Delta^n & \longrightarrow & Y \\ \big\downarrow & \nearrow & \\ \Delta^1 \times \Delta^n & & \end{array}$$

Important Lemma

Lemma (Cisinski, Thm. 3.5.8)

If Y is a quasicategory, then $\mathrm{eval}_1 \colon h(\Delta^1, Y) \to Y$ has the right lifting property against monomorphisms that induce a bijection on objects.

Proof.

We get a lift in the last diagram via analyzing via lifting a few inner horns and one outer horn, using Joyal's lifting theorem. Since

$$\begin{array}{ccc} \mathrm{Fun}^{\simeq}(\Delta^n, Y) & \longrightarrow & \mathrm{Fun}(\Delta^n, Y) \\ \downarrow & & \downarrow \\ \mathrm{Fun}^{\simeq}(\partial\Delta^n, Y) & \longrightarrow & \mathrm{Fun}(\partial\Delta^n, Y) \end{array}$$

is a pullback, the lift lives in $\mathrm{Fun}^{\simeq}(\Delta^n, Y)_1$, solving the original problem.

Proof of $\mathrm{Fun}^{\simeq}(X,Y) = \mathrm{Fun}(X,Y)^{\simeq}$

For the non-obvious inclusion (that is $\subseteq$), it suffices to show that $\mathrm{Fun}^{\simeq}(X,Y)$ is a Kan complex because $\mathrm{Fun}(X,Y)^{\simeq}$ is the biggest Kan complex in $\mathrm{Fun}(X,Y)$.
We may restrict ourselves to proving it to be a left Kan complex because one can just apply $(\_)^{\mathrm{op}}$ and repeat the argument. So consider the lifting problem

$$\begin{array}{ccccc} \Delta^1 \times \partial\Delta^n \cup \{1\} \times \Delta^n & \longrightarrow & \mathrm{Fun}^{\simeq}(X,Y) & \longrightarrow & \mathrm{Fun}(X,Y) \\ \big\downarrow & & & & \\ \Delta^1 \times \Delta^n & & & & \end{array}$$

that induces

$$\begin{array}{ccccc} X \times \partial\Delta^n & \longrightarrow & h(\Delta^1, Y) & \longrightarrow & \mathrm{Fun}(\Delta^1, Y) \\ \big\downarrow & & \big\downarrow \scriptstyle{\mathrm{eval}_1} & \swarrow \scriptstyle{\mathrm{eval}_1} & \\ X \times \Delta^n & \longrightarrow & Y & & \end{array}$$

and lifts from bottom left to top right correspond to each other. If $n = 0$, the original problem is trivial. Hence assume $n > 0$ and get a lift in the bottom left square by the Important Lemma.

Proof of $\mathrm{Fun}^{\simeq}(X, Y) = \mathrm{Fun}(X, Y)^{\simeq}$

The lift $l \colon \Delta^1 \times \Delta^n \to \mathrm{Fun}(X, Y)$ factors through $\mathrm{Fun}^{\simeq}(X, Y)$ if it only hits isomorphism after evaluating at any object $x \in X_0$. Consider the diagram

$$\begin{array}{ccccc} l_{0,0}(x) & \longrightarrow & \cdots & \longrightarrow & l_{0,n}(x) \\ \downarrow & & & & \downarrow \\ l_{1,0}(x) & \longrightarrow & \cdots & \longrightarrow & l_{1,n}(x). \end{array}$$

Proof of $\mathrm{Fun}^{\simeq}(X, Y) = \mathrm{Fun}(X, Y)^{\simeq}$

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$$\begin{array}{ccccc} l_{0,0}(x) & \longrightarrow & \cdots & \longrightarrow & l_{0,n}(x) \\ \simeq\downarrow & & & & \downarrow\simeq \\ l_{1,0}(x) & \longrightarrow & \cdots & \longrightarrow & l_{1,n}(x). \end{array}$$

The vertical arrows are isomorphisms since the restriction of l to $\Delta^1 \times \partial\Delta^n$ factors over $\mathrm{Fun}^{\simeq}(X, Y)$.

Proof of $\mathrm{Fun}^{\simeq}(X, Y) = \mathrm{Fun}(X, Y)^{\simeq}$

The lift $l\colon \Delta^1 \times \Delta^n \to \mathrm{Fun}(X, Y)$ factors through $\mathrm{Fun}^{\simeq}(X, Y)$ if it only hits isomorphism after evaluating at any object $x \in X_0$. Consider the diagram

$$\begin{array}{ccccc} l_{0,0}(x) & \longrightarrow & \cdots & \longrightarrow & l_{0,n}(x) \\ \downarrow & & & & \downarrow \\ l_{1,0}(x) & \xrightarrow{\simeq} & \cdots & \xrightarrow{\simeq} & l_{1,n}(x). \end{array}$$

The horizontal arrows are isomorphisms since the restriction of l to $\{1\} \times \Delta^n$ factors over $\mathrm{Fun}^{\simeq}(X, Y)$.

Proof of $\mathrm{Fun}^{\simeq}(X, Y) = \mathrm{Fun}(X, Y)^{\simeq}$

The lift $l\colon \Delta^1 \times \Delta^n \to \mathrm{Fun}(X, Y)$ factors through $\mathrm{Fun}^{\simeq}(X, Y)$ if it only hits isomorphism after evaluating at any object $x \in X_0$. Consider the diagram

$$\begin{array}{ccccc} l_{0,0}(x) & \xrightarrow{\simeq} & \cdots & \xrightarrow{\simeq} & l_{0,n}(x) \\ \simeq\big\downarrow & & & & \big\downarrow\simeq \\ l_{1,0}(x) & \xrightarrow{\simeq} & \cdots & \xrightarrow{\simeq} & l_{1,n}(x). \end{array}$$

Hence all arrows are isomorphisms by 2-out-of-3. □