Kan Fibrations and the Kan-Quillen Model Structure

Higher-Category Theory Reading Seminar

IMPA

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Plan

Kan-complexes

- Review of saturated classes
- Anodyne extensions
- A bit of digression

Simplicial Homotopy

- Function complexes
- Left (Right) Homotopy
- Covering Homotopy Extension property

Kan-Quillen Model Structure

- Minimal fibration
- The key theorem and outline of the proof

- Kan's Ex^∞ functor
  - Some properties
- Inner Anodyne Extensions

A class $\mathcal{A}$ of monos in sSet is said to be saturated if

(1) $\mathcal{A}$ contains all the isos

(2) $\mathcal{A}$ is closed under pushouts i.e.

if
$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & A' \arrow[d, "i'"] \\ B \arrow[r] & B' \end{tikzcd}$$
is a pushout diagram and $i \in \mathcal{A}$ $\Rightarrow i' \in \mathcal{A}$

(3) $\mathcal{A}$ is closed under retracts i.e.

If
$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"] \arrow[rr, bend left, "id_A"] & B \arrow[r] \arrow[d, "i'"] & A \arrow[d, "i"] \\
C \arrow[r] \arrow[rr, bend right, "id_C"'] & D \arrow[r] & C
\end{tikzcd}$$
is a comm. diagram and $i' \in A$ $\Rightarrow i \in A$

(4) A is closed under coproducts

i.e. if $(X_k \xrightarrow{i_k} Y_k \mid k \in K)$

is a family of monos with $i_k \in A$ $\forall k \in K$

$$\sum_{k \in K} i_k : \sum_{k \in K} X_k \longrightarrow \sum_{k \in K} Y_k \text{ is in } A$$

(5) A is closed and ω-complete i.e. if

$(X_n \longrightarrow X_{n+1} \mid n = 1, \dots)$ is a countable

family of morphisms in A. Then

$$u_1 : X_1 \longrightarrow \varinjlim_{n \geq 1} X_n \text{ is also in } A$$

The intersection of all saturated classes containing a set of morphism say F is called saturated class generated by F.

If $n : A \longrightarrow X$ is an arbitrary mono in sSets, then.

$$
\begin{tikzcd}
\coprod_{e(X-A)_n} \partial\Delta^n \arrow[r] \arrow[d] & \coprod_{e(X-A)_n} \Delta^n \arrow[d] \\
sk^{n-1}(X)\cup A \arrow[r] & sk^n(X)\cup A
\end{tikzcd}
$$

$n \geq 0$

$e(X-A)_n$ is the set of non-deg n-simplices of X which aren't in A

$$X = \varinjlim_{n\geq -1} \left(sk^n(X) \cup A \right) \quad \text{and}$$

$$u_{-1} : sk^{-1}(X) \cup A \longrightarrow \varinjlim \left(sk^n(X) \cup A \right)$$

$n \geq -1$

$n: A \longrightarrow \cancel{A}$.

The saturated class generated by the family

$\{ \partial \Delta^n \longrightarrow \Delta^n \mid n \geq 0 \}$ is the class of all mono.

Def: The saturated class generated by the family

$$\{ \Lambda^n_k \longrightarrow \Delta^n \mid 0 \leq k \leq n, n \geq 1 \}$$

is called the class of anodyne extensions, denoted by <u>Ano</u>

Remark If $M \subseteq \mathrm{Mor}(\mathrm{sSet})$ then $\ell(M)$ is saturated

Prop. The following saturated classes coincide.

(1) $B :=$ { the set of all inclusions of
$\Delta^1 \times \partial\Delta^n \cup \{\varepsilon\} \times \Delta^n \longrightarrow \Delta^1 \times \Delta^n$
$, n \geq 0, \varepsilon \in \{0,1\}$ }

(2) $C :=$ { the set of all inclusions of
$\Delta^1 \times Y \cup \{\varepsilon\} \times X \longrightarrow \Delta^1 \times X$
$X \in sSet$
Y is a subcomplex of X
$\varepsilon \in \{0,1\}$ }

(3) An

Def: A map $p: A \to X$ is a Kan fibration
iff it has rlp wrt all anodyne extensions.

Thm: Any map $g: A \to B$ in sSet has a factorisation of form $g = pi$, where i: is an anodyne extension and p is a fib.

$$\begin{tikzcd} A \arrow[rr, "g"] \arrow[dr, "i"'] & & B \\ & E \arrow[ur, "p"'] & \end{tikzcd}$$

Proof: We've

The set L of comm. diagrams

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \Delta^n \arrow[r] & Y \end{tikzcd}$$

$n \geq 1$

$p: X \to Y$

$$
\begin{tikzcd}
\coprod \Lambda^n_k \arrow[r] \arrow[d, "j"'] & X \arrow[d, "p"] \\
\coprod \Delta^n \arrow[r] & Y
\end{tikzcd}
$$

$$
\begin{tikzcd}
\coprod \Lambda^n_k \arrow[r] \arrow[d, "j"'] & X \arrow[d, "j^0"'] \arrow[dr, dashed, "j"] & \\
\coprod \Delta^n \arrow[r] \arrow[rr, dashed, bend right] & X' \arrow[r, dashed] & Y
\end{tikzcd}
$$

$$\begin{tikzcd} X \arrow[rr, "j^0"] \arrow[dr, "l"'] & & X^1 \arrow[dl, "l^1"] \\ & Y & \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, "j^0"] \arrow[dr, "l"'] & X^1 \arrow[r, "j^1"] \arrow[d, "l^1"] & X^2 \arrow[dl, "l^2"] \\ & Y & \end{tikzcd}$$

with j^1 anodyne.

let $X^0 = X$, $l^0 = l$ $\quad E = \varinjlim_{n \geq 0} X^n$

let $p : E \longrightarrow Y$ be the map induced by l^n on each X^n

$$\begin{tikzcd} X \arrow[rr, "j"] \arrow[dr, "r"'] & & E \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$

j is anodyne.

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "h"] \arrow[d, hook] & E \arrow[d, "p"] \\ \Delta^n \arrow[r] & Y \end{tikzcd}$$

$\leftarrow$ has a diagonal filler

As Λ^n_k has finitely many non-degen. simplices we can factor through X^n for some $n \geq 0$

$$
\begin{tikzcd}
\Lambda^n_k \arrow[r, "h"] \arrow[d] & X^n \arrow[r, "j^n"] \arrow[d, "f^n"] & X^{n+1} \arrow[dl, "f^{n+1}"] \\
\Delta^n \arrow[r] \arrow[urr, dashed] & Y &
\end{tikzcd}
$$

$\square$

Corollary: $j : X \to B$ is anodyne iff it has llp w.r.t. the class of all fibrations.

Thm: Any map $f : X \to Y$ of sSet admits the following factorisation

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\ & E \arrow[ur, "p"'] & \end{tikzcd}$$

i is a mono
p is an acyclic fib.

Proof: Use the family $\{\partial\Delta^n \longrightarrow \Delta^n \mid n \geq 0\}$ instead of $\{\Lambda^n_k \longrightarrow \Delta^n \mid 0 \leq k \leq n,\ n \geq 1\}$ in the proof of prev. thm.

Def: $X \in sSet$ is called a Kan complex if $X \longrightarrow *$ is a fibration.

Example: (1) Sing. simp. complex of topological space.

(ii) $M \in \underline{Gpd}$, NM is a Kan complex.

<u>Lemma</u> If $p : A \to B$ is a fib, then $\exists$ a surjection $q : C \to B$, such that in the following pullback square

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "p'"'] & A \arrow[d, "p"] \\ C \arrow[r, "q"'] & B \end{tikzcd}$$

if p' is a fib (an acyclic fib) then p is a fib (an acyclic fib)

A digression

Def: A bundle with fibre F in sSet is a map $\varphi: E \to X$ s.t. $\forall$ n-simplices $\Delta^n \to X$ of X $\exists$ an iso λ

$$
\begin{tikzcd}
\Delta^n \times F \arrow[rr, "\lambda"] \arrow[dr, "\pi_1"'] & & \Delta^n \times_X E \arrow[dl, "\pi_1"] \\
& \Delta^n &
\end{tikzcd}
$$

e.g. $Y = \coprod_{\Delta^n \to X} \Delta^n$, then the map $Y \to X$ is a surjection.

since $\pi_1 : Y \times F \longrightarrow Y$ is a fibration

when F is a Kan complex, it follows

from the prev. lemma that a bundle

with Kan fiber F is a fibration.

Ex If G is simplicial group.

$\Rightarrow$ $p : E \longrightarrow X$ is a principal

G-bundle, then p is a bundle with

fibre G and thus a fibration.

Function complex

For $X, Y \in C^{\Delta^{op}}$ (C has small coprod.)

the function compl h is a sSet Item(A, B)

with set of n-simplices being the set of

maps $\Delta^n \otimes A \longrightarrow B$

$$\downarrow$$

$V.$

copowering of simp. objs. of sSet.

$[\mathrm{Hom}(a \otimes b, c)$ $\cong \mathrm{Hom}_{\otimes V}(a, [b, c])$ $\longrightarrow$ Int. hom.

$V \to$ closed monoidal category.

$C \to V$-enriched cat.

copower of $c \in C$ by $k \in V$

is $k \odot c \in C$ with

$$C(K \odot T, Y) \cong V(K, [T, Y])$$

Simplicial Homotopy

Def For $X \in sSet$, $X \times \Delta^1$ the simp. cylinder obj.

Then a left htpy $\ell : f \Rightarrow g$ between $f, g : X \to Y$ is a morph

$$\ell : X \times \Delta^1 \longrightarrow Y$$

s.t. the following comm.

$$\begin{tikzcd}
X \arrow[d, "(id_X, d_1)"'] \arrow[dr, "f"] & \\
X \times \Delta^1 \arrow[r, "\ell"] & Y
\end{tikzcd}$$

$$f \overset{\ell}{\simeq} g$$

(id_X, d_0) $\quad$ X $\quad$ g

For a Kan complex X, its simp. path obj is the function complex X^{Δ^1}

Def A right htpy $r: f \Rightarrow g$ between $f, g: X \to Y$ is a morphism $r: X \longrightarrow Y^{\Delta^1}$ s.t. the follow diag. commute

$$\begin{tikzcd}
 & & Y \\
X \arrow[urr, "f"] \arrow[rr, "h"] \arrow[drr, "g"'] & & Y^{\Delta^1} \arrow[u] \arrow[d] \\
 & & Y
\end{tikzcd}
\qquad f \overset{r}{\sim} g$$

$$f \sim g \iff (f \overset{l}{\sim} g) \wedge (f \overset{r}{\sim} g)$$

Def.: A morphism $f: X \to Y$ is a left (right) htpy equiv. if $\exists$ a morphism $g: Y \to X$ and left (right)

Homotopy

$$g \circ f \Rightarrow id_X \qquad f \circ g \Rightarrow id_Y$$

Claim: Y is Kan, then Y^X is also Kan

for $X, Y \in sSet$.

Let $u : Y \longrightarrow Z$ be a mono and

$p : E \longrightarrow X$

$$\begin{tikzcd}
(u, p, \ldots) \arrow[r] \arrow[d] & E^Y \arrow[d, "p^Y"] \\
X^Z \arrow[r, "X^u"'] & X^Y
\end{tikzcd}$$

(k,p) is the obj of diag. of form

$$
\begin{tikzcd}
Y \arrow[r] \arrow[d, "k"'] & E \arrow[d, "p"] \\
Z \arrow[r] & X
\end{tikzcd}
$$

$$
\begin{tikzcd}
E^Z \arrow[r, "E^k"] \arrow[d, "p^Z"'] & E^Y \arrow[d, "p^Y"] \\
X^Z \arrow[r, "X^k"'] & X^Y
\end{tikzcd}
$$

$k \pitchfork p : E^Z \longrightarrow (k,p)$

Thm If p is a fib. then K/p is a fib. and K/p is acyclic if K is anodyne or p is acyclic

Corollary If $p: E \longrightarrow X$ is a fib so is

$$p^Y : E^Y \longrightarrow X^Y \text{ for any } Y.$$

Notation For monos $j: A \to B$, $u: Y \to Z$ we denote the inclusion

$$(A \times Z) \cup (B \times Y) \longrightarrow B \times Z$$

by $j \star u$.

Claim: If j is anodyne so is $j \hat{\times} K$

Covering Homotopy Extension Property (CHEP) for fib.

Proof: Let $q : F \to Y$ be a fib and $h : Z \times I \to Y$ a homotopy. Suppose $X \to Z$ is a map and $h' : X \times I \to F$ is a lifting of h to F on $X \times I$:-

$$\begin{tikzcd}
X \times I \arrow[r, "h'"] \arrow[d] & F \arrow[d, "q"] \\
Z \times I \arrow[r, "h"'] & Y
\end{tikzcd}$$

Suppose $\tilde{h} : Z \times (\varepsilon) \to F$ via lift of h_ε $(\varepsilon = 0, 1)$ to E

$$\begin{tikzcd}
Z \times (\varepsilon) \arrow[r] \arrow[d] & F \arrow[d, "q"] \\
Z \times I \arrow[r, "h"'] & Y
\end{tikzcd}$$

Then $\exists$ a homotopy $\bar{h} : Z \times I \to E$ which lift h i.e. $p\bar{h} = h$

pf:

$$\begin{tikzcd}
(X \times I) \cup (Z \times (\varepsilon)) \arrow[r] \arrow[d] & E \arrow[d, "q"] \\
Z \times I \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

$(\varepsilon) \to I$ is anodyne

$A \to B$ is a mono in sSet

Thm If $j : A \to B$ is anodyne. & A, B are Kan complex, then A is a strong deformation retraction of B

of if $j: A \to B$ is mono. s.t. A is a strong deformation retract of B, then j is anodyne.

Thm If K is a Kan complex then $p: A \to K$ is acyclic if p is a htpy equiv.

Def: A sset map $q: A \to B$ is said to be a minimal fib if q is a fib of Kan

fo

$$\begin{tikzcd}
\partial\Delta^n \times \Delta^1 \arrow[r, "f"] \arrow[d, hook] & \partial\Delta^n \arrow[d] \\
\Delta^n \times \Delta^1 \arrow[r, "h"] \arrow[d] & A \arrow[d] \\
\Delta^n \arrow[r] & B
\end{tikzcd}$$

$$\Delta^n \underset{d^1}{\overset{d^0}{\rightrightarrows}} \Delta^n \times \Delta^1 \text{ by } g$$

x, y in A_n are fibrewise homotopic (relative $\partial\Delta^n$) then $x = y$.

We denote this by $x \sim y \mod \partial\Delta^n$

Thm: X is a Kan complex. $\exists$ a strong def. retract X' of X which is minimal.

Def: Let $f: X \longrightarrow Y$ be a map in sSet.

f is weak equiv if $\forall$ Kan complexes K

$[f, K]: [Y, K] \longrightarrow [X, K]$ is a bijection.

$g, f: X \longrightarrow Y$, then $f \sim g$ if $\pi_0(f) = \pi_0(g)$

so if f is a htpy equiv, $\pi_0(f)$ is a biject

Also, $\beta f \sim \beta g$ for any β.

If f is a htpy equiv & K is a Kan, then

K^f is a htpy equiv & $\pi_0(K^f) = [f, K]$

is a biject.

If $X \longrightarrow \bar{X}$, $Y \longrightarrow \bar{Y}$ are anodyne extensions with $\bar{X}, \bar{Y}$ Kan, and $\bar{f}$ to be a lift

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & \bar{X} \arrow[d, dashed, "\bar{f}"] \\ Y \arrow[r] & \bar{Y} \end{tikzcd}$$

$\bar{f}$ is a htpy equiv.

$\boxed{\text{Key Thm}}$

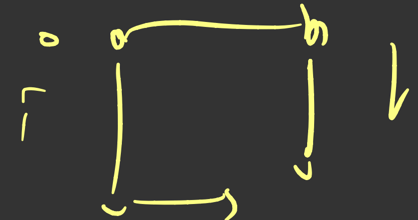
In sSet if we set Fib as Kan fibrations, Cofib as monos and weak equiv as defined then we get a proper Quillen ~~htpy~~ model structure.

<u>Model cat axioms</u>

M1 → 3-for-2 property.

M2 → Closure under retract.

M3 → lifting



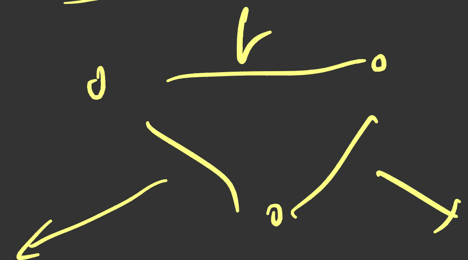
$i \in$ Cofib

$p \in$ Fib

If i or $p \in W$

$\Rightarrow$ a lift

M4 Factorization



(Acyclic cofib, fib)

or

(cofib , acyclic fib)

M5

$\Rightarrow w' \in W$

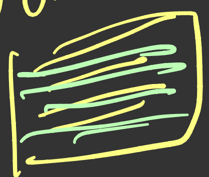
Lemma (1) Let $A \to B$ be an anodyne exten. of $p: E \to A$ bundle. Then $\exists$ a pullback s.th.

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ A \arrow[r] & B \end{tikzcd}$$

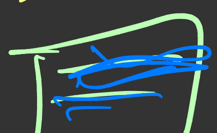
s.t. $p': E' \to B$ is a bundle and $E \to E'$ is anodyne

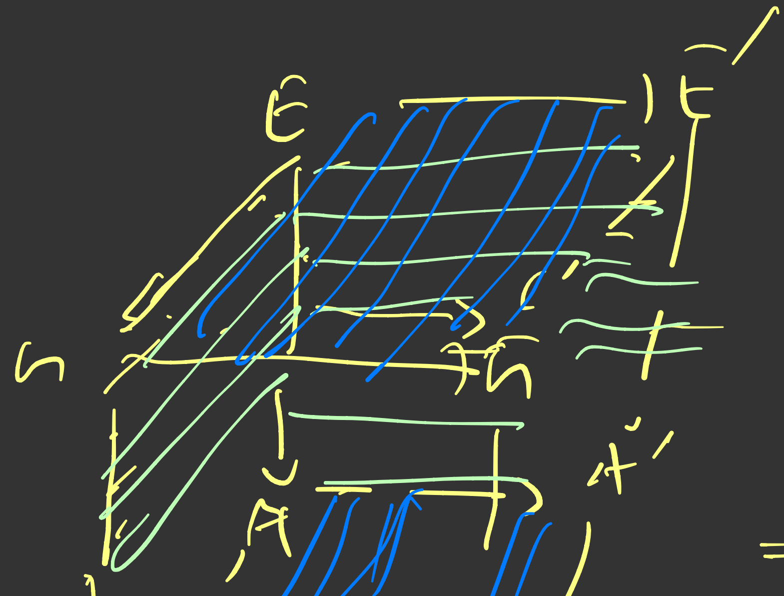
Lemma (2)

pullback



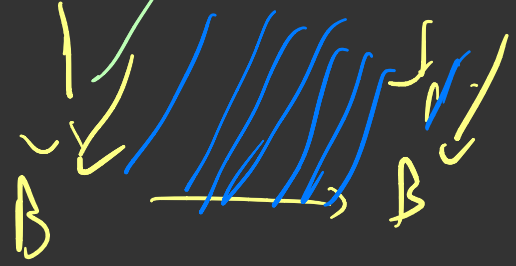
pushout





union cube is s fib.

if $A \to B$ is a mono $\Rightarrow$ right & bunt



faces are pullback

Proof: μ_1, μ_2 are clear. μ_3 holds by defn.

φ_5 needs a bit of work (minimal fib)

φ_4 by prev. result

Kan's Ex^∞-functor

$$Ex^\infty X \longrightarrow X$$

The non-deg simp. of Δ^n form $P\Delta^n$.

$$P\Delta^n \cong [n]$$

$$sd\,\Delta^n = NP\Delta^n$$

Def: For $X \in sSet$

$$sd\,X = \operatorname*{colim}_{\Delta^n \to X} sd\,\Delta^n$$

subdivn of Δ^2

$$\begin{tikzcd}
& 1 \arrow[dl] \arrow[d] \arrow[dr] & \\
01 \arrow[r] & 012 & 12 \arrow[l] \\
0 \arrow[r] \arrow[u] & 02 \arrow[u] & 2 \arrow[l] \arrow[u]
\end{tikzcd}$$

Def: For $X \in sSet$ we define $Ex(X)$ by

$$Ex(X)_n = \mathrm{Hom}_{sSet}(sd\,\Delta^n, X)$$

Obs. $Z_n \cong \mathrm{Hom}_{sSet}(\Delta^n, Z)$ for $Z \in sSet$.

$Ex: sSet \longrightarrow sSet$ is right adj. to sd

Def: The last vertex map

$\ell v: sd\,\Delta^n \longrightarrow \Delta^n$ is the map induced by the maps of posets $A\Delta^n \to [n]$

$$(i_0, \ldots, i_m) \longmapsto i_m$$

$X \in sSet$. This extends to give a map
$\hat{j} : X \longrightarrow Ex(X)$

Adj to Id is the map $\hat{j} : X \longrightarrow Ex(X)$

$$X \xrightarrow{\hat{j}_X} Ex(X) \xrightarrow{\hat{j}_{Ex(X)}} Ex^2(X) \xrightarrow{\hat{j}_{Ex^2(X)}} \cdots$$

We denote the colimit by $Ex^\infty(X)$

<u>Thm</u> $Ex^\infty(X)$ is a Kan complex for $X \in sSet$

<u>Proof</u> For any $\lambda : \Lambda^n_k \longrightarrow Ex(X)$
$\to$ an exten-

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "\lambda"] \arrow[d, hook] & Ex(X) \arrow[d, "j_{Ex(X)}"] \\
\Delta^n \arrow[r, dashed] & Ex^2(X)
\end{tikzcd}$$

Inner Anodyne Extensions

Def: The closure of the saturated class generated by inner horn inclusion.

Prop: The following class of morphisms coincide

B_1 = {class of inner anodyne ext}

$B_2 = \{$ smallest saturated class generated by

$$\Delta^2 \times \partial\Delta^n \cup \Lambda^2_1 \times \Delta^n \longrightarrow \Delta^2 \times \Delta^n$$

for $n \geq 0 \}$

$B_3 = \{$ smallest sat. class generated by

$$\Delta^2 \times K \cup \Delta^2_1 \times L \longrightarrow \Delta^2 \times L$$

for any mono

$K \to L \}$

Def: A map p is said to be an inner fib. if p has the r.l.p. w.r.t. the saturated class generated by inner horn inclusions

Claim These morph with llp until all mirror links are mirror-antisym.

Thanks