2.1 Factorization Systems

Def 2.1

left lifting property

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

"i has LLP wrt p"

"i has LLP wrt a class of morphisms F"

↳ $\ell(F)$ is class of morphisms w/ LLP against F

right lifting property

Same diagram, but

"p has RLP wrt i"

etc.

Def 2.1.2

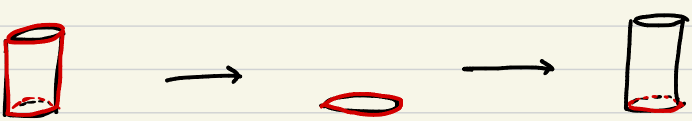
retract

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[rr, bend left, "1_X"] & U \arrow[r, "p"] & X \end{tikzcd}$$

"X is a retract of U"

ex

Non ex



- morphisms can be viewed as retracts in the category of morphisms

For a class of morphisms F in $\mathcal{C}$,

- stable under retracts
  - ↳ retracts of things in F are in F

- stable under pushouts

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "(po)"] & X' \arrow[d, "f'"] \\ Y \arrow[r] & Y' \end{tikzcd}$$

↳ f in F $\Rightarrow$ f' in F

- stable under transfinite completion

· I
  - a well ordered set, initial element 0

· $X : I \to \mathcal{C}$
  - functor st $\varinjlim_{j<i} X(j)$ representable for all nonzero $i \in I$

and

$$\varinjlim_{j<i} X(j) \to X(i)$$

is in F

Stable under transfinite compositions means

$$\varinjlim_{i \in I} X(i)$$

exists and the canonical

$$X(0) \xrightarrow{\in F} \varinjlim_{i \in I} X(i)$$

- saturated
  ↳ stable under
  - retracts
  - pushouts
  - transfinite compositions

Prop 2.1.4 ... nice things happen if $\ell(F)$ is saturated

Prop 2.1.5 Retract lemma.

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & T \arrow[d, "p"] \\ Y \arrow[r, equal] & Y \end{tikzcd}$$

" f has LLP wrt p
$\Rightarrow$ f is retract of i "

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "i"'] & X \arrow[d, "f"] \\ T \arrow[r, "p"'] & Y \end{tikzcd}$$

" f has RLP wrt i
$\Rightarrow$ f is a retract of p "

$*$ Def 2.1.7

A **weak factorization system** in $\mathcal{C}$ is two classes of morphisms (A, B) st

- A, B are stable under retracts
- $A \subset \ell(B)$

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "\in A"'] & \cdot \arrow[d, "\in B"] \\ \cdot \arrow[r] \arrow[ur, dashed, "\exists"] & \cdot \end{tikzcd}$$

- any morphism in C factors through A and then B

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\in A"'] & & Y \\ & \cdot \arrow[ur, "\in B"'] & \end{tikzcd}$$

$*$ Prop 2.1.9

Small object argument

If $\mathcal{C}$ is a category with nice properties*, and $I \subset \mathrm{Mor}(\mathcal{C})^+$, then every morphism f factors as

$$\begin{tikzcd} \cdot \arrow[rr, "f"] \arrow[dr, "\in \mathrm{Cell}(I)"'] & & \cdot \\ & \cdot \arrow[ur, "\in r(I)"'] & \end{tikzcd}$$

$*$, $+$: can relax condition $(*)$ but need to tighten condition $(+)$

$\mathrm{Cell}(I)$ = transfinite compositions of pushouts (think cell-complexes from AT...)

Cor 2.1.10

A : small category
I : small set of morphisms of presheaves over A

$\Rightarrow$ $(l(r(I)), r(I))$ is a WFS in $\hat{A}$.

Prop 2.1.12 "Compatibility of WFS's w/ adjunctions"

"compatible one way implies compatible the other way too"

$$\begin{tikzcd}
\mathcal{C} \arrow[r, shift left, "F"] & \mathcal{C}' \arrow[l, shift left, "G"] \\
\begin{pmatrix} A \\ B \end{pmatrix} \arrow[u, "\text{WFS}"] \arrow[r, dashed, shift left] & \begin{pmatrix} A' \\ B' \end{pmatrix} \arrow[u, "\text{WFS}"'] \arrow[l, dashed, shift left]
\end{tikzcd}$$

(adjunction)

$$F(A) \subset A' \iff G(B') \subset B$$

2.2 Model Categories

Def 2.2.1 A **model category** $\mathcal{C}$ is a locally small category w/ 3 classes of morphisms

W — "weak equivalences"

Fib — "fibrations"

$Cofib$ — "cofibrations"

st

1. $\mathcal{C}$ has finite limits and colimits
2. W satisfies "2 out of 3" property

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

3. Both

$(Cof, Fib \cap W)$

$(Cof \cap W, Fib)$

are WFS's.

$Fib \cap W$ — "trivial / acyclic fibrations"

$Cofib \cap W$ — "trivial / acyclic cofibrations"

$X \in Obj(\mathcal{C})$ is **fibrant** if $X \longrightarrow e$ is in Fib
(e ← final object)

similarly, is **cofibrant** if $\emptyset \longrightarrow X$ is in $Cofib$
($\emptyset$ ← initial object)

$\emptyset \xrightarrow{cof} QX \longrightarrow X$

Rmk — **Weak equivalences** are really the core of the theory.

Fib/Cofib are like tools that, when they exist, can help us study WEs.

Ex 2.2.4 — Any appropriate category, taking W to be isomorphisms

Ex 2.2.5 — $\hat{A}$ for a small category A

$W =$ all morphisms

Cofib $=$ monomorphisms

Ex — Top, where $W =$ weak homotopy equivalence

Fib $=$ Serre fibrations

Cofib $=$ retract of relative cell complexes

Prop 2.2.6 — The following constructions (naturally) preserve the model structure:

- $\mathcal{C} \longrightarrow \mathcal{C}^{op}$

- for $X \in Obj(\mathcal{C})$, $\mathcal{C}/X$

We could ask how these 3 classes are related.

- If a functor does something to one class, do we know what it does to the others?

Prop 2.2.7 Ken Brown's lemma

$$F : \mathcal{C} \longrightarrow D$$

model category

has a class of weak equivalences

If $F(\mathrm{Cofib} \cap W) \subset W$ $(*)$

$\Rightarrow F(W) \subset W$

$*$ In fact, if

$F($Cofib $\cap W$ between cofibrant objects$) \subset W$

$\Rightarrow F(W) \subset W$

Def 2.2.8 Let W be a class of morphisms in $\mathcal{C}$. Localization (of $\mathcal{C}$ by W) is a functor

$$\gamma : \mathcal{C} \longrightarrow ho(\mathcal{C})$$

- universal one that sends W to isomorphisms

Prop 2.2.9

- There always exists a localization of $\mathcal{C}$ by any class W
- We can strengthen the universal property: can pick γ s.t.

$$\gamma^*: \underline{\mathrm{Hom}}(\mathrm{ho}(\mathcal{C}), D) \longrightarrow \underline{\mathrm{Hom}}_W(\mathcal{C}, D)$$

↑ functors sending W to isomorphisms

is not simply an equivalence of categories, but an isomorphism.

Construction $\mathrm{ho}(\mathcal{C})$

- Objects are same as $\mathcal{C}$
- Morphisms are diagrams of the form

$$\begin{tikzcd}
& X_1 \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & X_n \arrow[dl] \arrow[dr] & \\
X = X_0 & & X_2 & & X_{n-1} & & X_{n+1} = Y
\end{tikzcd}$$

where each arrow is in W or an isomorphism.

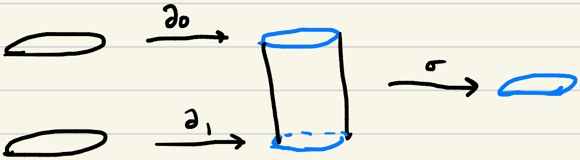
$$\begin{tikzcd}
& X_1 \arrow[dl] \arrow[r] \arrow[dd] & X_2 \arrow[dd] & \cdots \arrow[l] \arrow[r] & X_{n-1} \arrow[dd] & X_n \arrow[l] \arrow[dd] \arrow[dr] & \\
X & & & \cdots & & & Y \\
& X_1' \arrow[ul] \arrow[r] & X_2' & \cdots \arrow[l] \arrow[r] & X'_{n-1} & X_n' \arrow[l] \arrow[ur] &
\end{tikzcd}$$

Basically pretending things in W are invertible.

Fix a model category $\mathcal{C}$.

Def 2.2.11 **Cylinder**

$$\begin{tikzcd} A \sqcup A \arrow[r, "(\partial_0, \partial_1) \in \mathrm{Cofib}"] \arrow[rr, bend right, "(1_A, 1_A)"'] & IA \arrow[r, "\sigma \in W"] & A \end{tikzcd}$$



Cocylinder / path object

$$\begin{tikzcd} X \arrow[r, "s \in W"] \arrow[rr, bend right, "(1_X, 1_X)"'] & X^I \arrow[r, "(d^0, d^1) \in \mathrm{Fib}"] & X \times X \end{tikzcd}$$

Left homotopy $f_0 \Rightarrow_L f_1$ $\qquad f_0, f_1 : A \longrightarrow X$

$$\begin{tikzcd} A \arrow[d] \arrow[rrd, bend left, "f_0"] & & \\ A \sqcup A \arrow[r, "(\partial_0, \partial_1)"'] & IA \arrow[r, "h"'] & X \\ A \arrow[u] \arrow[rru, bend right, "f_1"'] & & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} A \arrow[d, "\partial_0 i"'] \arrow[rd, bend left, "f_0"] & \\ IA \arrow[r, "h"'] & X \\ A \arrow[u, "\partial_1 i"] \arrow[ru, bend right, "f_1"'] & \end{tikzcd}$$

Lemma 2.2.12 $\exists$ left homotopy $\Longleftrightarrow$ $\exists$ right homotopy

<u>if</u> A is cofibrant, X is fibrant

Lemma 2.2.13 left (right) homotopy is an equivalence relation.
↳ A cofibrant, X fibrant

write
$$[A, X] = \mathrm{Hom}_{\mathcal{C}}(A, X) / \sim$$

- Left homotopy is compatible w/ composition on the left:

$$\begin{tikzcd}
Z \arrow[r, "g"] & A \arrow[d, "\partial_0"'] \arrow[dr, bend left, "f_0"] & \\
& IA \arrow[r, "h"'] & X \\
Z \arrow[r, "g"'] & A \arrow[u, "\partial_1"] \arrow[ur, bend right, "f_1"'] &
\end{tikzcd}$$

... right homotopy compatible on the right

$$\rightsquigarrow \quad [-,-] : \mathcal{C}_c^{op} \times \mathcal{C}_f \longrightarrow \mathrm{Set}$$

Thm 2.2.15
$$\mathrm{ho}(\mathcal{C}_c) \simeq \mathrm{ho}(\mathcal{C}) \qquad \text{(equivalence of categories)}$$
$$\mathrm{ho}(\mathcal{C}_f) \simeq \mathrm{ho}(\mathcal{C})$$

Prop 2.2.16 There is a natural extension of the functor
$$[-,-] : \mathcal{C}_c^{op} \times \mathcal{C}_f \longrightarrow \mathrm{Set}$$
to
$$[-,-] : \mathrm{ho}(\mathcal{C}_c^{op}) \times \mathrm{ho}(\mathcal{C}_f) \longrightarrow \mathrm{Set}$$

Thm 2.2.17

Recall, for $A \in \mathcal{C}_c$ and $X \in \mathcal{C}_f$,

$$[A, X] := \hom_{\mathcal{C}}(A, X) / \sim$$

$\uparrow$ left homotopy

The claim is there is a bijection

$$[A, X] \simeq \mathrm{Hom}_{ho(\mathcal{C})}(A, X)$$

... natural wrt morphisms of $ho(\mathcal{C}_c^{op}) \times ho(\mathcal{C}_f)$.

- morphisms in the first case are mod left homotopy, and in the second case are mod weak equivalences!

Cor 2.2.18

Define $\pi(\mathcal{C}_{cf})$ as the category with

- objects are fibrant-cofibrant objects in $\mathcal{C}$
- $\mathrm{Hom}_{\pi(\mathcal{C}_{cf})}(A, X) = [A, X]$

Then

$$\pi(\mathcal{C}_{cf}) \simeq ho(\mathcal{C})$$

is a (canonical) equiv. of categories.

$$\rightsquigarrow \quad \begin{aligned} ho(\mathcal{C}_c) &\simeq ho(\mathcal{C}) \\ ho(\mathcal{C}_f) &\simeq ho(\mathcal{C}) \\ \pi(\mathcal{C}_{cf}) &\simeq ho(\mathcal{C}) \end{aligned}$$

- the relationship between objects in $\mathcal{C}$ and weak equivalences is the same as

    - $\rightsquigarrow$ cofibrant objects and weak equivalences between them
    - $\rightsquigarrow$ similarly fibrant objects
    - $\rightsquigarrow$ fibrant-cofibrant objects and <u>left homotopy</u>

$$D^{n} \sqcup_{S^{n-1}} D^n = S^n$$

$$* \sqcup_{S^{n-1}} *$$

2.3 Derived functors

How compatible are functors with localization? $\rightsquigarrow$ the tool for "approximating" functors like this is Kan extensions

Def 2.3.1
$\mathcal{C}$ - model category
$\gamma : \mathcal{C} \longrightarrow ho(\mathcal{C})$ - localization
$F : \mathcal{C} \longrightarrow D$ - a functor

left derived functor

$$LF : ho(\mathcal{C}) \longrightarrow D$$

together with

$$a_X : LF(\gamma(X)) \longrightarrow F(X)$$

such that LF is a right Kan extension along γ.

Dually for right derived functors.

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "\gamma"'] & \arrow[d, phantom, "\Uparrow a"] & D \\ & ho(\mathcal{C}) \arrow[ur, "LF"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "\gamma"'] & \arrow[d, phantom, "\Downarrow b"] & D \\ & ho(\mathcal{C}) \arrow[ur, "RF"'] & \end{tikzcd}$$

Q: Do these exist?
↳ general theory on existence of Kan extensions

What do they look like?

Consider the situation of Ken Brown's lemma.

$$F : \mathcal{C} \to \mathcal{D}$$

sends weak equivalences between cofibrant objects to isomorphisms.

By the universal property, $\exists!$

$$F_c : ho(\mathcal{C}_c) \longrightarrow \mathcal{D}$$

For each $X \in \mathcal{C}$, pick

$$a_X' : \underset{\text{Cof}}{QX} \xrightarrow{\text{GW}} X,$$

(which exists since $\emptyset \longrightarrow X$ must factor as $\emptyset \xrightarrow[\text{Cof}]{} QX \xrightarrow[\text{GW} \cap \text{Fib}]{} X$)

so this is a map $\mathcal{C} \longrightarrow \mathcal{C}_c$

$\rightsquigarrow$ $\exists!$ functor $Q : ho(\mathcal{C}) \longrightarrow ho(\mathcal{C}_c)$

Define LF:

$$LF(Y) = F_c(Q(Y))$$

$(\exists!)$ $\quad a_X : LF(\gamma(X)) \longrightarrow F(X)$

Prop 2.3.3 In the above situation, LF is a left derived functor of F.

Cor 2.3.4 In the above situation, for any functor

$$G : \mathcal{D} \longrightarrow \mathcal{E}$$

the pair $(G \circ LF, G\alpha)$ is a left derived functor of GF.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[dr, "\gamma"'] & \mathcal{D} \arrow[r, "G"] & \mathcal{E} \\ & ho(\mathcal{C}) \arrow[u, "LF"'] \arrow[ur, dashed, "G \circ LF"'] & \end{tikzcd}$$

Def 2.3.5 total derived functors

$\mathcal{C}, \mathcal{C}'$ - model categories
γ, γ' - respective localizations
F - functor $\mathcal{C} \to \mathcal{C}'$

If F preserves $W \cap Cof$, then

$$\mathcal{C} \xrightarrow{F} \mathcal{C}' \xrightarrow{\gamma'} ho(\mathcal{C}')$$

Satisfies Ken Brown conditions, so

$$\Rightarrow \exists! \quad ho(\mathcal{C}) \xrightarrow{LF} ho(\mathcal{C}')$$

total left derived functor of F = left derived functor of $\gamma' F$

similarly get total right derived functor of F

$$ho(\mathcal{C}) \xrightarrow{RF} ho(\mathcal{C}')$$

Prop 2.3.6 ("Well behaved-ness of total derived functors")

If $\mathcal{C} \xrightarrow{F} \mathcal{C}' \xrightarrow{F'} \mathcal{C}''$

both preserve cofibrant objects and WN Wol, then

$$\mathrm{ho}(\mathcal{C}) \xrightarrow{LF} \mathrm{ho}(\mathcal{C}') \xrightarrow{LF'} \mathrm{ho}(\mathcal{C}'')$$

and

$$\mathrm{ho}(\mathcal{C}) \xrightarrow{L(F' \circ F)} \mathrm{ho}(\mathcal{C}'')$$

are isomorphic on objects $X \in \mathrm{ho}(\mathcal{C})$.

What do adjunctions look like on the level of localization?

↳ Are they still adjunctions?

Def 2.3.7 Quillen adjunctions

An adjunction between model categories

$$F : \mathcal{C} \rightleftarrows \mathcal{C}' : G$$

that descends to an adjunction

$$LF : \mathrm{ho}(\mathcal{C}) \rightleftarrows \mathrm{ho}(\mathcal{C}') : RG$$

Thm If F preserves cofibrations, G preserves fibrations, then (F, G) is a Quillen adjunction.

↳ (In the above situation)

Rmk 2.3.8 ("duality" of fibrations / cofibrations across adjunctions)

TFAE:

- (F, G) is a Quillen adjunction
- F preserves cofib. and $W \cap \text{Cof}$.
- G preserves fib. and $W \cap \text{Fib}$

Why? Want to get homotopy-analogs of useful functors.

Def Projective model structure (on $\mathcal{C}^I := \text{Hom}(I, \mathcal{C})$)

<u>If</u> it exists, then $F \longrightarrow G$ is a fibration if $F_i \longrightarrow G_i$ is a fibration for all $i \in I$.

These don't necessarily have to exist.
So... when do they exist?

Ex $\quad I$ is a small, discrete category

Ex $\quad I$ is free category generated by $0 \longrightarrow 1$

(Prop 2.3.11) $\quad \rightsquigarrow \mathcal{C}^I$ is the arrow category of $\mathcal{C}$

Ex $\quad \mathcal{C}$ has small colimits

(Prop 2.3.13) $\quad I$ is small, well-ordered, initial object

Ex $\quad$ A projective resolution of a chain complex Q is a cofibrant resolution $\emptyset \to PQ \longrightarrow Q$ wrt the projective model structure on chain complexes.

Why are these called projective model categories?

- whenever one exists,

$$\begin{tikzcd} \mathcal{C}^I \arrow[r, shift left, "\varinjlim_I"] & \mathcal{C} \arrow[l, shift left, "\delta"] \end{tikzcd}$$

constant diagram

is a <u>Quillen adjunction</u>.

Prop 2.3.15 Suppose

- $\mathcal{C}^I$ has proj. model structure
- $\mathcal{C}$ has I-indexed colimits

If $f : X \longrightarrow Y$ ($X \in \mathcal{C}^I$, $Y \in \mathcal{C}^I$) B cofibrant

and every

$$f_i : X_i \longrightarrow Y_i \quad \in W,$$

then

$$\varinjlim_I X \longrightarrow \varinjlim_I Y \quad \in W.$$

Cor 2.3.16 - 2.3.19 Spefic cases of these

Def 2.3.20

F - functor $I \to \mathcal{C}$
X - object in $\mathcal{C}$
X_I - constant diagram $I \to \mathcal{C}$

Consider a cocone under F.

$$\text{“}\left\{ \begin{tikzcd} Fi \arrow[r] \arrow[dr] & Fj \arrow[d] \\ & X \end{tikzcd} \right\}\text{”}$$

X is the homotopy colimit of F if the induced

$$(\mathbb{L}\varinjlim_I)(F) \xrightarrow{\sim} X$$

is an isomorphism.

- $\varinjlim_I : \mathcal{C}^I \to \mathcal{C}$
- $\mathbb{L}\varinjlim_I : ho(\mathcal{C}^I) \to ho(\mathcal{C})$

Ex 2.3.21 These <u>sometimes</u> coincide w/ ordinary colimits.

- If $\mathcal{C}$ has I-indexed colimits, has a projective model structure, and F is cofibrant, ~~then~~ then

$$\varinjlim_{I} F$$

is the homotopy colimit of F

Def 2.3.22 homotopy cocartesian / homotopy pushout

$$\begin{tikzcd} X \arrow[r, "x"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, "y"'] & Y' \end{tikzcd}$$

where Y' is homotopy colimit of $Y \leftarrow X \rightarrow X'$

Ex (another instance of regular and homotopy colimits coinciding)

If a pushout is between cofibrant objects, with cofibrations, then it is also a homotopy pushout.

Prop 2.3.26 Any commutative square

$$\begin{tikzcd} X \arrow[r, "x"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, "y"'] & Y' \end{tikzcd}$$

in which $x, y \in W$ is homotopy cocartesian.

Cor 2.3.28 In the above square, if it is coCartesian, $f \in \mathrm{Cof}$, X and X' cofibrant, then it is homotopy coCartesian.