A 2-categorical Quillen's Theorem A

Part 1

(Work by Garcia and Stern)

Higher Category Reading Seminar, htpy
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Plan

- Idea and the motivation
- Basic 2-cat thy
  - Intro
  - Duskin Nerve
  - Lax slices and functoriality
- ∞-localisation

Thm (Quillen Thm A)

$\mathcal{C}, \mathcal{D} \in \mathrm{Cat}$ and $F : \mathcal{C} \to \mathcal{D}$.
$\forall$ objects x of D the geom. realisation $|N(F_{/x})|$ of the comma cat. F/x is contractible then

$$|N(F)| : |N(\mathcal{C})| \longrightarrow |N(\mathcal{D})| \text{ is a weak htpy equ.}$$

A version of Quillen's Thm A

Thm $X, Y \in \mathrm{Cat}$, $F : X \to Y$ is a functor
Then if $\forall$ $x \in Y$
$\exists$ $x' \in X$ and a map $x \to F(x')$
and every morphism $x \to F(z)$ represent an initial obj of $N(\mathcal{C}_{x/})^{\triangleleft}$
Then $|F| : |N(\mathcal{C})| \longrightarrow |N(\mathcal{D})|$ is a htpy equiv.

2-cat thy

Abuse of notation: 2-Cat - means a strict 2-cat.

Notn $X \in 2\mathrm{Cat}$

1-morph is denoted $X(op, -)$
2-morph " $X(-, op)$
1,2-morph " $X(op, op)$

Def. $X, Y \in 2Cat$. A normal lax 2-funct. $F: X \to Y$

- a map $F: Ob(X) \longrightarrow Ob(Y)$
- $\forall$ pairs $x, x' \in X$, a functor
$$F_{x,x'} : X(x,x') \longrightarrow Y(F(x), F(x'))$$
- $\forall$ triple $x, x', x'' \in X$,

$$\begin{tikzcd}
X(x',x'') \times X(x,x') \arrow[rr, "\circ"] \arrow[d, "F"'] & \arrow[d, phantom, "\swarrow \sigma"] & X(x,x'') \arrow[d, "F"] \\
Y(F(x'), F(x'')) \times Y(F(x), F(x')) \arrow[rr, "\circ"] & & Y(F(x), F(x''))
\end{tikzcd}$$

$\exists$ a nat'l tran. σ (compositor)

s.t.
- $F_{x,x}(Id_x) = Id_{F(x)}$ $\forall x \in X$
- the compositor satisfies the hexagon identity
- $\sigma_{f,Id} = Id_{F(f)}$ & $\sigma_{Id,f} = Id_{F(f)}$

$\underline{LCat}$ → cat. of 2-cats with normal lax 2-fun.

Def: We get a cosimp. obj. in $LCat$ by composing $\Delta^\bullet : \Delta \to Cat$ with $j: Cat \to LCat$
(Abuse of notation: we denote $j\Delta^\bullet$ by $\Delta^\bullet$)

Then we get a functor using this cosimp. obj.
$N_2 : LCat \to sSet$ with $N_2(X)_n = LCat([n], X)$
(Duskin Nerve)

Def: Let I be a linearly ord. fin. set, then let's define O^I.
obj: elements of I
- The cat $O^I(x,y)$ of morph $x, y \in I$ is defined as a poset of fin. sets $S \subseteq I$ s.t. $\min(S) = x$, $\max(S) = y$
- The comp. is given by $x, y, z \in I$
$$O^I(x,y) \times O^I(y,z) \longrightarrow O^I(x,z)$$

(Notat. $O^{[n]} = O^n$)

Cosimpl. obj $O^\bullet$

The composite $\Delta \xrightarrow{O^\bullet} 2Cat \to LCat$ is denoted by $O^\bullet$!

$\forall n \in N_2$, we can define
$$\xi_n : [n] \to O^n$$ defn on 1-mor $\xi_n(x \le y) = \{x, y\}$

Prop: $X \in 2Cat$
(1) N_2 is fully faithful
(2) The lax functors ξ_n form a nat'l trans. of cosimp obj.
(3) $\forall n \in N_2$ the lax functors ξ_n induce a bijection
$$LCat([n], X) \cong 2Cat(O^n, X)$$

Lax slices and functoriality

Def. $X \in 2\mathrm{Cat}$ and $x \in X$.
Let's define $X_{x/}^{\vee} \in 2\mathrm{Cat}$, the lax slice cat, as follows.

Objs of $X_{x/}^{\vee}$ are morphisms $f : x \to y$ in X and

Morphisms are from $f : x \to y$, $g : x \to z$ are consisting of a morphism $u : y \to z$ in X with a 2-morph
$\beta : g \Rightarrow u \circ f$.

A 2-morphism from (u, β), (v, α) consists of 2-morph $\eta : u \Rightarrow v$ s.t.

$$\begin{tikzcd} g \arrow[r, Rightarrow, "\beta"] \arrow[dr, Rightarrow] & u \circ f \arrow[d, Rightarrow] \\ & v \circ f \end{tikzcd}$$

commute.

We can define the oplax slice cat $X_{x/}$ as

$$\left(\left(X^{(-,op)} \right)_{x/}^{\vee} \right)^{(-,op)}$$

Prop. $2\mathrm{Cat}_* \longrightarrow \mathrm{Cat}$, $(X, x) \longmapsto X_{x/}^{\vee}$ is a functor.

∞-bicategories

Def. A marked 2-cat is a pair $X^\dagger = (X, W_X)$ where $X \in 2\mathrm{Cat}$ and $W_X \subseteq \mathrm{Mor}(X)$ containing all identities.

A functor of marked 2-cats $F : X^\dagger \to Y^\dagger$ is a 2-functor $F : X \to Y$ s.t. $F(W_X) \subset W_Y$.

Notation: $2\mathrm{Cat}^\dagger$ = cat of marked 2-cats.

Def. A cat with weqs is a pair $X^+ = (X, W_X)$ where $X \in \mathrm{Cat}$ and $W_X \subset X$ is a wide subcat.
A homotopical functor $F : X^+ \to Y^+$ is a functor $F : X \to Y$ s.t. it maps W_X to W_Y.

Analogously a 2cat with weqs Y^+ consists of a 2cat Y with the str. of a cat with weqs on the underlying 1-cat.
Similarly we define a homotopical 2-functor.

Notation: $2\mathrm{Cat}^{we}$ = cat of 2-cats with weqs and homotopical 2-functors.

Def: A marked 2-cat (X, W_X) is called saturated if

1) W_X contains all equiv of X

2) (X, W_X) is a 2-cat with weq.

3) For $f \in W_X$, $g \in X$ & we've an invertible 2-morph $f \overset{\simeq}{\Rightarrow} g$ then $g \in W_X$

Def: For a fibration $f : X^+ \longrightarrow Y^+$ of marked 2-cats, $\forall\, y \in Y^+$, we can define a marked 2-cat X_y^+ whose underlying 2-cat is the lax slice cat with an edge marked iff the assoc. 2-morph is invertible & the assoc. 1-morph is marked in X^+.

Def. $(X, W) \in sSet^+$. An (∞-cat) localisation of X by W is a ∞-cat $L_W(X)$ equipped with map $\gamma_X : X \longrightarrow L_W(X)$ in $sSet^+$, s.t. $\forall\, \mathcal{C} \in \infty Cat$, the map

$$\gamma_X^* : \mathrm{Fun}((X, W), \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}(L_W(X), \mathcal{C})$$

is an equiv of ∞-cat.

Corollary 7.1.1.0 $\mathrm{Fun}(X, W) \in \infty sSet^+$, the loc.

$$\gamma_X : X \longrightarrow L_W(X) \text{ is cofinal \& invertible.}$$

[cofinal $f : X \longrightarrow Y$ st. precomp. with f preserves ∞-colim]

Thank you