Categories fibered in ∞-groupoids

$\mathcal{C} : Cat_0 \quad \hat{\mathcal{C}} \hookrightarrow Cat/_{\mathcal{C}}$

$\mathcal{F} : \hat{\mathcal{C}}_0$ consider $\mathcal{C}/_{\mathcal{F}} \to \mathcal{C}$

$\hat{\mathcal{C}} \xhookrightarrow{El} Cat/_{\mathcal{C}} \quad \mathcal{F} \mapsto (\mathcal{C}/_{\mathcal{F}} \to \mathcal{C})$

$\varphi : \mathcal{I} \to \mathcal{C}$

$\mathcal{I} \xrightarrow[\varphi]{} \mathcal{C} \xrightarrow[h]{} \hat{\mathcal{C}}$

colim $(h \circ \varphi)$ = left adjoint of El on φ.

$\varphi : \mathcal{I} \to \mathcal{C}$

$R(\varphi : \mathcal{I} \to \mathcal{C})(c) = \{ u : \mathcal{C}/_c \to \mathcal{I} \mid$
$\varphi u : \mathcal{C}/_c \to C$ is the can. projection$\}$

$Cat/_{\mathcal{C}}$

$\mathcal{C}/_{\mathcal{F}} \to \mathcal{C}$ these are the

functors $p : X \to \mathcal{C}$ s.t. $\forall x : X_0$
$c = p(x)$ $X/_x \longrightarrow \mathcal{C}/_c$ is an iso.

Prop (4.1.2) $p : X \to \mathcal{C} : Set_1^{\Delta^{op}}$ is
a RFib $\iff$ p : InnFib s.t. $\forall x : X_0$
if we put $c = p(x)$ the induced f-r

$X/x \longrightarrow \mathcal{C}/c$ is a trivial fib.

<u>Proof:</u> $\Lambda^n_n = \partial\Delta^{n-1} * \Delta^0 \quad (n>0)$

apply (L. 3.4.20)

$$\begin{tikzcd} \partial\Delta^{n-1} \arrow[r] \arrow[d, hook] & X/x \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{C}/c \end{tikzcd} \iff \begin{tikzcd} \Lambda^n_n \arrow[r, "u"] \arrow[d, hook] & X \arrow[d] \\ \Delta^n \arrow[r, "v"'] \arrow[ur, dashed] & \mathcal{C} \end{tikzcd}$$

$u(n) = x$

$\mathrm{RFib} = \mathrm{Inn\,Fib} + r(\Lambda^n_n \hookrightarrow \Delta^n)$

$\square$

<u>Cor (4.1.3)</u> The nerve of any G. fib w/ discrete fibres between small cats is a right fibration.

<u>Proof:</u> $N(\mathcal{C})/x \cong N(\mathcal{C}/x)$

now we can apply the previous prop. to the characterization of G. fib. in simpl. terms. $\square$

Assume that $\mathcal{C} : \mathrm{Set}^{\triangledown}_0$. $P(\mathcal{C})$ is a full subcategory of $\mathrm{Set}^{\triangledown}/\mathcal{C}$ w/ objects being the right fibrations of the form

$p_{\mathcal{F}} : \mathcal{F} \to \mathcal{C}$.

$P(\mathcal{C})_0$ = "right fibrant over $\mathcal{C}$"

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "f"] \arrow[dr] & & \mathcal{G} \arrow[dl] \\ & \mathcal{C} & \end{tikzcd} \quad : P(\mathcal{C})_1$$

f is a fiberwise equivalence if $\forall c : \mathcal{C}_0$ the induced map $\mathcal{F}_c \xrightarrow{\sim} \mathcal{G}_c$ is an equiv. of ∞-groupoids.

$$\begin{tikzcd} \mathcal{F}_c \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{F} \arrow[d, "p_{\mathcal{F}}"] \\ \Delta^0 \arrow[r, "c"'] & \mathcal{C} \end{tikzcd}$$

right fibration

by Cr 3.5.6

$\mathcal{F}_c : \infty$ Gpd.

Theorem (Joyal) (4.1.5)

There is a unique model category str-re on the category $Set^{\Delta^{op}}/\mathcal{C}$ with

$Cof = Mono(Set^{\Delta^{op}}/\mathcal{C})$

$(Set^{\Delta^{op}}/\mathcal{C})_f = P(\mathcal{C})$

Moreover, $Fib \cap P(\mathcal{C})_1 = RFib$

Observation: If $\mathcal{C} = \Delta^0 \Rightarrow$ the contravariant model str-re is just the K-Q model structure.

$\mathcal{C}$: ∞-Gpd $\Rightarrow$ CV m.s. is induced by the projection $Set^{\Delta^{op}}/_{\mathcal{C}} \to Set^{\Delta^{op}}$ from K-Q m.s.

$J' = N(0 \rightleftarrows 1)$ it is Kan complex (∞-gpd + all morphisms are invertible)

$J' \to \Delta^0$ is a simplicial homotopy equivalence.

$J' \times X \longrightarrow X$ it is trivial fib. $\forall X : Set^{\Delta^{op}}_0$

J' defines an exact cylinder $X \mapsto J' \times X$

$An^{r}_{J'}$ J'-anodyne extensions which contains $\Lambda^n_k \hookrightarrow \Delta^n$ $n \geq 1$ $0 < k \leq n$

By L. 3.1.3. $An^{r}_{J'}$ is the minimal sat. class containing the following maps:

1. $J' \times \partial\Delta^n \cup \{\varepsilon\} \times \Delta^n \to J' \times \Delta^n$ $n \geq 0$ $\varepsilon = 0, 1$.
2. $\Lambda^n_k \hookrightarrow \Delta^n$ $n \geq 1$ $0 < k \leq n$.

Prop (4.1.7) $An^r_{J'}$ is just the class of right anodyne extensions.

Proof: Any right anodyne ext. is contained in $An^r_{J'}$.

We want to show that $\forall\, i$: Generators of $An^r_{J'}$ we have $i \boxslash RFib$

$$i : A \hookrightarrow B = N(\mathcal{C}_B) \quad : \mathrm{Cat}_0$$

Let $p : X \to Y$: RFib

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, hook, "i"'] & X \arrow[d, "p"] \\
B \arrow[r, "b"'] \arrow[ur, dashed] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, hook, "i"'] & B \times_Y X \arrow[r, "\beta"] \arrow[d, "q"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\
B \arrow[r, equal] & B \arrow[r, "b"] & Y
\end{tikzcd}$$

q : RFib + codom$(q) = B : \infty\mathrm{Cat}$

Now we can assume that codom$(p) : \infty\mathrm{Cat}$

We must prove that $\forall\ p : X \to Y$ $p : \mathrm{RFib}$, $Y : \infty\mathrm{Cat}$ the evaluation map at $\varepsilon = 0, 1$ is a triv. fib:

$$\underline{\mathrm{Hom}}(J', X) \longrightarrow X \times_Y \underline{\mathrm{Hom}}(J', Y)$$

Observe that p is always an isofibration (by $\underline{\text{Prop } 3.4.8}$) $\Rightarrow$ $p : \mathrm{Fib}_{\mathrm{Joyal}}$ ($\underline{\text{Thm } 3.6.1}$) Since $J' : \infty\mathrm{Gpd}$ we have $h(J', C) = \underline{\mathrm{Hom}}(J', C)$ $\forall$ $C : \infty\mathrm{Cat}$ $\Rightarrow$ we can apply a result from §3 which yields an existence of the lift. $\square$

$\underline{\text{Proof of Thm } 4.1.5}$ The theorem now becomes an instant of the construction from § 2.5 due to Prop 4.1.7 $\Rightarrow$ we can apply the theorem 2.4.19 for

$$A = \Delta/C \qquad \hat{A} = \hat{\Delta}/C \qquad \square$$

$\underline{\text{Def } 4.1.8}$ $u : A \to B : \mathrm{Set}_\Delta^\nabla$ it is **final** if $\forall\ C : \mathrm{Set}_\Delta^\nabla$ and any $p : B \to C$ $u : (A, pu) \to (B, p)$

is a weak equivalence in cv-m.s. on $Set^{\nabla}/_{\mathcal{C}}$.

Corollary 4.1.9. f : Mono(Set^{∇})
f is right anodyne extension iff it
it is a final map. $f : Set^{\nabla}_{1}$ is
final iff $f = pi$, $p : Fib^{+}$, $i : An^{r}$

The class $\mathbb{C}$ of final maps is the
smallest class in Set^{∇}_{1} s.t.:
(a) $\mathbb{C}$ is closed under composition
(b) $\forall\ f : X \to Y$, $g : Y \to Z$ if
$f, gf : \mathbb{C} \Rightarrow g : \mathbb{C}$

(c) $An^{r} \subset \mathbb{C}$

This immediately follows from Prop. 2.5.3, 4

Corollary (4.1.10) $f : X \to Y$, $g : Y \to Z$
: Mono (Set^{∇}). $f : An^{r} \Rightarrow$
$g : An^{r} \Leftrightarrow gf : An^{r}$

Prop (4.1.11) $f : X \to Y$: Set^{∇}_{1}
$p : Y \to \mathcal{C}$: RFib. Then f is final $\Leftrightarrow$
$\Leftrightarrow$ p turns f into W of cv m.s.

Let $p_{\mathcal{F}} : \mathcal{F} \to \mathcal{C}$, $p_{\mathcal{G}} : \mathcal{G} \to \mathcal{C}$

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}}(\mathcal{F},\mathcal{G}) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F},\mathcal{G}) \arrow[d, "(p_G)_*"] \\ \Delta^0 \arrow[r, "p_{\mathcal{F}}"'] & \underline{\mathrm{Hom}}(\mathcal{F},\mathcal{C}) \end{tikzcd}$$

If G is right fibrant over $\mathcal{C}$ $\Rightarrow$

$\mathrm{Map}_{\mathcal{C}}(\mathcal{F},\mathcal{G})$ is a fiber of $(p_G)_*$: RFib

$\Rightarrow$ (by prop. 3.4.5) it is a Kan complex

$c : \mathcal{C}_0$ $\quad h(c)$ the image of the corresponding morphism $\Delta^0 \xrightarrow{c} \mathcal{C}$

The canonical iso $\underline{\mathrm{Hom}}(\Delta^0, X) \cong X$

induces a canonical iso:

$$\mathrm{Map}_{\mathcal{C}}(h(c), G) \cong G_c$$

Prop 4.1.13 $\quad \begin{tikzcd} \mathcal{F} \arrow[rr, "f"] \arrow[dr] & & \mathcal{F}' \arrow[dl] \\ & \mathcal{C} & \end{tikzcd}$ and $G : P(\mathcal{C})_0$

$$\mathrm{Map}_{\mathcal{C}}(\mathcal{F}', \mathcal{G}) \xrightarrow{f^*} \mathrm{Map}_{\mathcal{C}}(\mathcal{F}, \mathcal{G})$$

is a Kan fibration.

Proof: $\mathrm{dom}(f^*)$, $\mathrm{codom}(f^*)$ are Kan complexes. By Prop. 3.5.5 it is sufficient to show that f^* is a right Kan fibration. Since RKFib are stable under base change (+ Prop 3.4.5) we have $\lrcorner$ of RKFib:

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}}(\mathcal{F}',\mathcal{G}) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathrm{Hom}}(\mathcal{F}',\mathcal{G}) \arrow[d, "\mathrm{RKFib}"] \\
\mathrm{Map}_{\mathcal{C}}(\mathcal{F},\mathcal{G}) \arrow[r, hook] & \underline{\mathrm{Hom}}(\mathcal{F}',\mathcal{G}) \times_{\underline{\mathrm{Hom}}(\mathcal{F}',\mathcal{C})} \underline{\mathrm{Hom}}(\mathcal{F},\mathcal{C})
\end{tikzcd}$$

□

Prop 4.1.14 $\forall\, f : \mathcal{F} \to \mathcal{F}' : W_{(\mathrm{Set}^{\Delta}/\mathcal{C})_{cv}}$ $\forall\, \mathcal{G} : P(\mathcal{C})_0$ the map

$$f^* : \mathrm{Map}_{\mathcal{C}}(\mathcal{F}',\mathcal{G}) \to \mathrm{Map}_{\mathcal{C}}(\mathcal{F},\mathcal{G})$$

is an equiv of ∞-Grpd.

Proof: Analogously to the previous prop one sees that $\forall\, \mathcal{G} : P(\mathcal{C})_0$ the f-r

$\mathrm{Map}_{\mathcal{C}}(-, G)$ sends An^r into trivial fibrations. By Prop. 2.4.40 and 4.1.7 $\Rightarrow$ $\mathrm{Map}_{\mathcal{C}}(-, G)$ is a left Quillen f-r from $(\mathrm{Set}^{\nabla}/\mathcal{C})^{op}_{CV} \to$

$\to (\mathrm{Set}^{\nabla})_{KQ}$.

In particular this f-r preserves W between cofibrant objects $\Rightarrow$ the prop holds. $\square$

<u>Lemma 4.1.15</u> $n : \mathbb{Z}_{\geq 0}$ $\{n\} \hookrightarrow \Delta^n$ $: \mathrm{An}^r$.

<u>Proof:</u> Let $h : \Delta^1 \times \Delta^n \to \Delta^n$ be defined by the formula:

$$(\varepsilon, x) \mapsto \begin{cases} n & , \varepsilon = 1 \\ x & , \text{otherwise} \end{cases}$$

Δ^n can be identified w/ $\{0\} \times \Delta^n$ $\Rightarrow$ we have the following diagram:

$$\begin{tikzcd}
\{n\} \arrow[r, hook] \arrow[d, hook] & \Delta^1 \times \{n\} \cup \{1\} \times \Delta^n \arrow[r] \arrow[d, hook] & \{n\} \arrow[d, hook] \\
\Delta^n \arrow[r, "\{0\} \times -"] & \Delta^1 \times \Delta^n \arrow[r, "h"] & \Delta^n
\end{tikzcd}$$

$\Rightarrow \{n\} \hookrightarrow \Delta^n$ is a retract of something in $An^?$ $\Rightarrow \{n\} \hookrightarrow \Delta^n : An^?$.

$\square$

<u>Theorem</u> (4.1.16) $\varphi : \mathcal{F} \to \mathcal{G} : P(\mathcal{C})_1$

TFAE:

(i) $\varphi : W_{(Set^{\Delta}/\mathcal{C})_\omega}$

(ii) φ is a fiberwise equivalence

(iii) $\forall X : (Set^{\Delta}/\mathcal{C})_0$ the induced map

$\varphi_* : Map_{\mathcal{C}}(X, \mathcal{F}) \to Map_{\mathcal{C}}(X, \mathcal{G})$

is an equiv of ∞Gpd.

<u>Proof:</u> Since $\mathcal{F}_c \simeq Map_{\mathcal{C}}(h(c), \mathcal{F})$ we see that (iii) $\Rightarrow$ (ii). We have a cartesian square for $Map_{\mathcal{C}}(X, -)$ $\Rightarrow$ this f-r preserves trivial fibrations $\Rightarrow$ it preserves $W_{P(\mathcal{C})}$ (i) $\Rightarrow$ (iii).

By prop 4.1.13,14 $\Rightarrow \forall X : (\mathrm{Set}^{\nabla}/_{\mathcal{C}})_0$

$\forall \mathcal{F} : P(\mathcal{C})_0$ there an identification of $\pi_0(\mathrm{Map}_{\mathcal{C}}(X, \mathcal{F}))$ and the set $[X, \mathcal{F}]$ because a cylinder over X is sent into a path object of $\mathrm{Map}_{\mathcal{C}}(X, \mathcal{F})$ in KQ m.s.

(iii) $\Rightarrow$ (i).

It now remains to show that (ii) $\Rightarrow$ (iii)

$n : \mathbb{Z}_{\geq 0}$ by L. 4.1.15, Prop. 4.1.14

$\forall s : \Delta^n \to \mathcal{C}$ we have a comm. $\square$:

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}}((\Delta^n, s), \mathcal{F}) \arrow[r, "\sim"] \arrow[d, "\varphi_*"] & \mathcal{F}_{s(n)} \arrow[d] \\
\mathrm{Map}_{\mathcal{C}}((\Delta^n, s), \mathcal{G}) \arrow[r, "\sim"] & \mathcal{G}_{s(n)}
\end{tikzcd}$$

(ii) $\Leftrightarrow$ $\forall s : \Delta^n \to \mathcal{C}$ the morphism φ_* is an equiv of ∞Gpd

$\mathrm{Map}_{\mathcal{C}}(-, \mathcal{F})$ send colimits in $\mathrm{Set}^{\nabla}/_{\mathcal{C}}$ into limits in Set^{∇}.

The class of simplicial sets $X : \mathrm{Set}^{\nabla}/_{\mathcal{C}}$ s.t. $\mathrm{Map}_{\mathcal{C}}(X, \mathcal{F}) \xrightarrow{\varphi_*} \mathrm{Map}_{\mathcal{C}}(X, \mathcal{G})$

is an equiv of ∞ Gpd $\to$ Set by monomorphisms. Apply Cor 1.3.10 for $A = \Delta/e$ to prove (ii) $\Rightarrow$ (iii) $\square$

Corollary: 4.1.17: The class W_{cv} is closed under filtered colimits.

Proof:

RFib are closed under filtered colimits

Fib res. f^{-1} commutes w/ them

Apply the previous theorem to reduce to the case of fiberwise w.e.

$\Rightarrow$ we only need to check that $W_{\infty Gpd}$ satisfy this property $\Rightarrow$ Cor 3.9.8 $\square$