05 November 2022 22:56

Homotopy Groups of Kan complexes and the Long Exact Sequence of Fibrations

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Amartya Shekhar Dubey

05-11-2022

Plan

- Homotopy Groups
- Pointed Kan complexes
- The connecting homomorphism and the LES of a fibration
- Fully faithful and Essentially surj. functor.

Def: The fundamental groupoid of a Kan complex X, is it's homotopy cat. $\mathrm{h}(X) = \pi_{\leq 1}(X)$

Def: For each $x \in X$ let's denote

$\mathrm{Aut}_{\pi_{\leq 1}(X)}(x)$ by $\pi_1(X, x)$, which is the fundamental group of

X (and x)

$\forall f: X \to Y$, we've induced maps

$$\pi_{\leq 1}(f): \pi_{\leq 1}(X) \longrightarrow \pi_{\leq 1}(Y)$$

Homotopy Groups

Def: X is a fib. sset and $x \in X$, we assoc a seq. of sets $\{\pi_n(X, x)\}_{n \geq 0}$.

- For $n \geq 1$ we define $\pi_n(X, x)$ is the set of htpy classes of maps $\alpha: \Delta^n \to X$ (rel $\partial\Delta^n$) that fit into

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d, hook] \arrow[dr, dashed, bend right=60] & \Delta^0 \arrow[d] \\
\Delta^n \arrow[r, "\alpha"] & X
\end{tikzcd}$$

- $\pi_0(X)$ is the set of htpy classes of vertices of X

Remark: (1) For $n = 1$ we get back the fundament group.

(2) $f: A \to B$ is a Kan fib is Kan

$a \in A$, and $f(a) = b \in B$ then we denote the fibre of f over the vertex b by $A_b = \{b\} \times_B A$

$$\cdots \to \pi_{n+1}(B,b) \xrightarrow{\partial} \pi_n(A_b, a) \longrightarrow \pi_n(A, a) \longrightarrow \pi_n(B,b) \to \pi_{n-1}(A_b, a) \to \cdots$$

Thm: $\pi_n(X,x)$ is a group for $n \geq 1$ (it's abelian for $n \geq 2$)

$$\left[\pi_n(X,x) \xrightarrow{\cong} \pi_{n-1}(\Omega X, x) \right]$$

(Pt = pointed)

Pointed Kan complex

Def: A pointed sSet is a pair (X,x) where $X \in sSet$, x is a vertex of X. If $X \in Kan$ then (X,x) is a pt Kan complex.

We've a categorical structure (Kan_*)

Objs → $(X,x), (Y,y), \ldots \in Kan_*$

Morph → $(X,x) \xrightarrow{f} (Y,y)$ is a map $f : X \to Y$, $f(x) = y$

(denoted as $[f : X \to Y]$)

Def: For $X \in sSet$, a simp. subset $Y \subseteq X$ is called a summand of X if $X = Y \sqcup Z$ for some $Z \subseteq X$ (simp. subset).

Def: For $X \in sSet$, X is connected if $X \neq \emptyset$, & the summand is either $\emptyset$ or X itself.

Def (Connected comp.)

For $X \in sSet$, a simp. subset $Y \subseteq X$ is called a conn. component if Y is a summand of X and Y is connected

Def (Pointed htpy cat.)

(X,x), $(Y,y) \in sSet_*$.

For pointed maps $f: X \longrightarrow Y$ $f \sim_* g$ (f is ~~pt~~ htpic) if they belong to the same comp. comp. of

$$\mathrm{Fun}(X,Y) \times_{\mathrm{Fun}(\{x\},Y)} \{y\}$$

Def A pt htpy from f to g is a morph.

$$h: \Delta^1 \times X \longrightarrow Y \quad \text{s.t.}$$

$$h|_{\{0\} \times X} = f \text{ and } h|_{\{1\} \times X} = g$$

Prop: For (X,x), $(Y,y) \in sSet_*$, $f,g: X \longrightarrow Y$, $f \sim_* g$ iff $\exists$ a seq. of pointed maps such $f = h_0, h_1, \dots, h_n = g$ st $\forall\ 1 \le i \le n$ $\exists$ a pt htpy from f_{i-1} to f_i (f_i to f_{i-1} or from)

Consq. If Y is Kan complex then $f \sim_* g$ iff $\exists$ a pt htpy from f to g.

Not. $[X,x]_* = \pi_0\left(\mathrm{Fun}(X,Y) \times_{\mathrm{Fun}(\{x\},Y)} \{y\} \right)$.

Htpy cats of pt Kan cmplx.

Thm The cat. $\mathrm{Ho}(\mathrm{Kan}_*)$ has the following data

Obj: pt Kan cmplx

Mor: $\mathrm{Hom}_{\mathrm{Ho}(\mathrm{Kan}_*)}\big((X,x),(Y,y)\big) = [X,Y]_*$

$[g]\circ[f] = [gf]$

– comp. law

$$\mathrm{Hom}_{\mathrm{Ho}(\mathrm{Kan}_*)}\big((Y,y),(Z,z)\big) \times \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Kan}_*)}\big((X,x),(Y,y)\big) \longrightarrow \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Kan}_*)}\big((X,x),(Z,z)\big)$$

Htpy of Kan cmplx

Def: $x \in X$ and $x_0 \in S^n$

the nth htpy group $\pi_n(X,x)$ denote the

set of htpy class of pt maps

$(S^n, x_0) \longrightarrow (X,x)$ endowed with a group str.

Notation A a set, for a non-empty subset $B \subseteq A$

$A/B = A \sqcup_B \{v\}$.

Rmk For $(X,x) \in$ Kan cmplx $n \in \mathbb{Z}^{\geq 0}$ ... $\Delta^n/\partial\Delta^n \longrightarrow X$

Rmk For $(X,v) \in$ Kan complex $n \in \mathbb{Z}^{\geq 0}$

Then the set of pt morphs $\Delta^n/\partial\Delta^n \longrightarrow X$

can be identified with the set of n-simplices

$x: \Delta^n \to X$ st. $x|_{\partial\Delta^n}$ is the const. map $\partial\Delta^n \to \{v\} \subseteq X$

— $[x] = \mathrm{Im}(x)$ in $\pi_n(X,x)$

Thm: Let $(X,v) \in \mathrm{Kan}_*$, $n \in \mathbb{Z}^{>0}$, then $\exists!$ group str. on $\pi_n(X,v)$

(a) For $e: \Delta^n \to \{v\} \to X$, $[e]$ is the id. element of $\pi_n(X,v)$

(b) Let $f: \partial\Delta^{n+1} \to X$ correspond to a tuple $(x_0, \ldots, x_{n+1})$ of n simplices of X

st. $x_i|_{\partial\Delta^n}$ is equal to $\partial\Delta^n \to \{v\} \subseteq X$ for $0 \leq i \leq n+1$

then f extends to $\Delta^{n+1} \longrightarrow X$ iff the prod.

$[x_0]^{-1}[x_1][x_2]^{-1}[x_3] \cdots [x_{n+1}]^{(-1)^{n}}$

is equal to id. elt of $\pi_n(X,v)$

Prp (functoriality)

$f: X \longrightarrow Y \in \mathrm{Mor}(\mathrm{Kan})$ and $y = f(v) \in Y$ for $v \in X$

$\forall\, n \geq 1$ well-def homomorphisms

$\pi_n(f): \pi_n(X,v) \longrightarrow \pi_n(Y,y)$

$$[X] \longmapsto [f(X)]$$

for $X : \Delta^n \to X$ with $X|_{\partial\Delta^n}$ being the const. map $\partial\Delta^n \to \{v\} \subseteq X$

$$\mathrm{Kan}_* \longrightarrow \mathrm{Grp}$$
$$(X, v) \longmapsto \pi_n(X, v)$$

<u>The connecting homomorphism</u>

Notation: $Y \in \mathrm{Kan}$ and $f : X \to Y$ is a Kan fib.
$X, Y \in sSet$

Fix $v \in X$ and $y = f(v)$. $X_y = \{y\} \times_Y X$.

Def: Let $f : X \to Y$ is a Kan fib. in Kan_* and $n \in \mathbb{Z}^{\geq 0}$.

If we've maps $b_0 : \Delta^n \to X_y$; $b_1 : \Delta^{n+1} \to Y$

s.t. $b_0|_{\partial\Delta^n}$, $b_1|_{\partial\Delta^{n+1}}$ are const. maps taking value v & y

Then b_0 is said to be incident to b_1 if $\exists\, \tilde{b}_1 : \Delta^{n+1} \to X$ with $b_1 = f(\tilde{b}_1)$ and $b_0 = d_0(\tilde{b}_1)$ and $\tilde{b}_1|_{\Lambda_0^{n+1}} : \Lambda_0^{n+1} \to X$ is the const. map taking value v

Prop: Let $f : X \to Y$ be a Kan fib in (Kan_*), $n \in \mathbb{Z}^{\geq 0}$

$\to \pi_n(X_y, v)$ ...

Lemma: Let $p: X \to Y$ ...

$$\exists!\, \partial : \pi_{n+1}(Y, y) \longrightarrow \pi_n(X_y, x) \quad \forall n.$$

if $b: \Delta^n \to X_y$ and $\tilde{b}: \Delta^{n+1} \to Y$ are simplices ~~with~~ s.t. $b_{|\partial\Delta^n}$, $\tilde{b}_{|\partial\Delta^{n+1}}$ are const maps taking value x only.

Then b is incident $\tilde{b}$ iff $\partial([\tilde{b}]) = [b]$

Def Let $p: X \to Y$ be a Kan fib in Kan_*. Then $\forall\, n \in \mathbb{Z}^{\geq 0}$, the map ∂ as defined is the conn. homomorphism.

Rmk For a Kan fib $p: X \to Y$ with $n \in \mathbb{Z}^{\geq 1}$, ∂ is a group homomorph.

The LES of a fib

Thm: Suppose $p: X \to Y$ is a Kan fib in Kan_*, then the seq of pt sets

$$\cdots \to \pi_2(Y, y) \xrightarrow{\partial} \pi_1(X_y, x) \longrightarrow \pi_1(X, x) \longrightarrow \pi_1(Y, y) \xrightarrow{\partial} \pi_0(X_y, x) \longrightarrow \pi_0(X, x) \longrightarrow \pi_0(Y, y)$$

is exact.

Proof. It suffices to check that the three both.

Proof Sketch It suffices to check that [illegible]

seq. are exact

$$\pi_n(X_y, x) \longrightarrow \pi_n(X, x) \longrightarrow \pi_n(Y, y) \quad (n \geq 0)$$

(1) $\pi_n(X_y, x) \longrightarrow \pi_n(X, x) \longrightarrow \pi_n(Y, y)$

(2) $\pi_{n+1}(Y, y) \xrightarrow{\partial} \pi_n(X_y, x) \longrightarrow \pi_n(X, x)$

(3) $\pi_{n+1}(X, x) \xrightarrow{\pi_{n+1}(f)} \pi_{n+1}(Y, y) \xrightarrow{\partial} \pi_n(X, x)$

(1) Let $x : \Delta^n \to X$ with $x|_{\partial\Delta^n}$ is constant map $\partial\Delta^n \to \{x\} \subseteq X$.
i.e.

ETS $[x] \in \mathrm{Im}(\pi_n(X_y, x) \longrightarrow \pi_n(X, x))$
iff $[f(x)]$ is the base pt of $\pi_n(Y, y)$

$(\Rightarrow)$ $X_y \longrightarrow X \xrightarrow{f} Y$

$(\Leftarrow)$ Suppose $[f(x)]$ is a basepoint of $\pi_n(Y, y)$

$\exists$ htpy $h : \Delta^1 \times \Delta^n \longrightarrow Y$ from $f(x)$ to $x_0' : \Delta^n \to \{y\} \subseteq Y$ which is const (restrict to $\partial\Delta^n$)

Since f is a Kan fib, we can lift to $\tilde{h} : \Delta^1 \times \Delta^n \to X$

from x to $x' : \Delta^n \longrightarrow X$ where x' is const

$\Rightarrow f(x') = x_0'$.

and $f(x') = x_0$.

(10)

Whitehead theorem for sets

Lemma $\pi_n(X, x) = \pi_1(\Omega^{n-1}(X), x)$

Prop.: For a weak htpy equivalence of sets $f : X \to Y$ and $x \in X$
if $y = f(x)$ then
$\pi_n(X,x) \to \pi_n(Y,y)$ is an iso for $n \geq 1$.

Pf: If $f : X \to Y$ is a weak htpy eq. then
iff $\pi_0(X) \simeq \pi_0(Y)$ and
$\pi_n(X,x) \to \pi_n(Y,y)$ is a iso for $n \geq 1$

Def.: For $F : X \to Y$, a functor between ∞-cats.
F is said to be fully faithful (ff) iff $\forall A, B \in X$

$$\mathrm{Hom}_X(A,B) \to \mathrm{Hom}_Y(F(A), F(B))$$

Remark: For a functor $F : A \to B$ between 1-cats,
F is ff iff $NF : NA \to NB$ is ff.

Def.: A functor $F : \mathcal{C} \to \mathcal{D}$ is ess. surj (of ∞-cats) iff
$\mathrm{Ho}\,F : \mathrm{Ho}\,\mathcal{C} \to \mathrm{Ho}\,\mathcal{D}$ is ess. surj.
$(\forall X \in \mathrm{Ho}\,\mathcal{D}\ \exists y \in \mathrm{Ho}(\mathcal{C})$ s.t.
iso $X \simeq \mathrm{Ho}(F(y)))$

$\simeq \operatorname{Ho}(F(U)))$

(iii) $\tilde{F} : \tilde{\mathcal{C}} \to \tilde{\mathcal{D}}$ is an ~~iso~~ ess. surj.

Thm. For a functor $F : \mathcal{C} \to \mathcal{D}$ between ∞-cat.

F is an eq. of ∞-cat iff F is ff & ess. surj.

Thanks