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The Joyal Model Structure

Higher Categories Reading Seminar
ntpy

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Plan

- A quick Recap
- Categorical Equivalences
- Dwyer Kan equivalences
- Pre-fibrant simplicial sets
- The Joyal Model Structure

Recall

We've the functor. $Ho: sSet \longrightarrow Cat$

$$S \longmapsto Ho(S)$$

Def: An edge $e: x \to y$ in an ∞-category $\mathcal{C}$ is called an equivalence if the image in $Ho(\mathcal{C})$ is an iso.

More generally we can replace $\mathcal{C}$ by an sSet S.

Def: Let $\mathcal{C}, \mathcal{D}$ be ∞-categories. A map $f : \mathcal{C} \to \mathcal{D}$ is an equivalence if $\exists$ a map $g : \mathcal{D} \to \mathcal{C}$ such that $\exists$ equivalences $gf \to id_{\mathcal{C}}$ and $fg \to id_{\mathcal{D}}$ in the ∞-cats $Fun(\mathcal{C},\mathcal{C})$ and $Fun(\mathcal{D},\mathcal{D})$.

<u>Notation</u> $f, g : A \to B$ in sSet are J-homotopic if $\exists$ a map $H : A \times J \to B$ such that $Hi_0 = f$ and $Hi_1 = g$ where $i_0, i_1 : A \to A \times J$.

J denotes the groupoid interval (groupoid with objects 0 and 1 and a unique iso between them)

<u>Remark</u> $f : \mathcal{C} \to \mathcal{D}$ is an equivalence of ∞-categories if it's a J-htpy equivalence in sense that $\exists$ a map $f' : \mathcal{D} \to \mathcal{C}$ such that fg, gf are J-homotopic to $id_{\mathcal{D}}$ and $id_{\mathcal{C}}$.

Lemma Let $f : X \to Y$ is a homotopy equivalence

<u>Lemma</u> Let $f : K \longrightarrow L$ is a homotopy equivalence where $K, L \in \underline{Kan}$. Then f is an equivalence of ∞-cat.

Categorical Equivalences

<u>Def</u> (Joyal) A map $f : A \to B$ of sSet is a cat. equivalence if for any ∞-cat Y, the map

$$\mathrm{Fun}(B, Y) \longrightarrow \mathrm{Fun}(A, Y) \text{ is an equivl}$$

of ∞-cats.

<u>Prop</u>: If $f : \mathcal{C} \to \mathcal{D}$ is an equivalence between ∞-cat, then f is a categorical equivalence.

<u>Lemma</u> If map $f : A \longrightarrow B$ is inner anodyne then f is a categorical equivalence

(HTT) 2.3._

<u>Lemma</u> If $g : X \longrightarrow Y$ is an acyclic fib. then g is a cat. equivalence

<u>Remark</u> The class of cat. equivalences in sSet is stable under filtered colimits.

Dwyer-Kan-equivalence (DK-equivalences)

($\mathcal{C}$ is an ∞ cat., $x, y \in \mathrm{Ob}(\mathcal{C})$) then $\exists$ a mapping space of morphisms $\mathrm{Hom}_{\mathcal{C}}(x, y)$ (well defined object in $\mathrm{Ho}(\mathrm{Sp})$)

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(left morphisms)

Def: $S \in \mathrm{sSet}$ and x, y are vertices of S, then the sSet $\mathrm{Hom}^L_S(x,y)$ of left morphisms from x to y in S is the sSet whose set of n-simplices is the set of all maps

$f : \Delta^{n+1} \to S$ with $f(0) = x$, $f|_{\Delta^{\{1,\dots,n\}}}$ is the constant n-simplices on vertex y.

Remark: If $\mathcal{C}$ is an ∞-cat then $\mathrm{Hom}^L_{\mathcal{C}}(x,y)$ is a Kan complex presenting $\mathrm{Map}_{\mathcal{C}}(x,y)$

S present $\mathcal{C}$ $\Rightarrow$ $\exists$ a cat. equiv $S \to \mathcal{C}$.

$\mathrm{Hom}^L_S(x,y) \in \mathrm{sSet}$

Def (DK equivalence): A, B : ∞-cats, A map $g : A \to B$ is

(i) essentially surjective if $\mathrm{Ho}(g) : \mathrm{Ho}(A) \to \mathrm{Ho}(B)$ is ess. surj.

(ii) fully faithful if

$$\mathrm{Hom}^L_A(k,k') \to \mathrm{Hom}^L_B(g(k), g(k'))$$

for any vertices $k, k' \in A$

If f is fully faithful & ess. surj., then f is a Dwyer-Kan equivalence.

... ∞-cats is an equiv iff it's a DK equiv

Claim: A map between ∞-cats is an equiv iff it is a 2N equiv

Lemma: If $f : A \to B$ is an equiv between ∞-cat A, B, then f is fully faithful.

Def: An inner fib. between ∞-cats X, Y $f : X \to Y$ is a cat. fib if $Ho(f) : Ho(X) \to Ho(Y)$.

Remark: An inner fib $f: A \to B$ is a cat. fib if it has rlp wrt $\{0\} \subseteq J$

Lemma (Joyal) If $p : A \to B$ is a cat. fib.. between ∞-cat A, B is fully faithful and essentially surj. then p is an acyclic Kan. fib.

Proof: Suppose p is fully faithful & ess. surj.

p has rlp wrt $\{0\} \subseteq J$,

We prove that for $n \geq 1$ the

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "u"] \arrow[d] & A \arrow[d] \\ \Delta^n \arrow[r, "v"] & B \end{tikzcd}$$

$$
\begin{tikzcd}
\partial \Delta^{\{1,\dots,n\}} \arrow[r] \arrow[d] & A_{v(0)/} \arrow[d] \\
\Delta^{\{1,\dots,n\}} \arrow[r] \arrow[ur, dashed] & B_{v(0)/} \times_B A
\end{tikzcd}
$$

The induced map $A_{v(0)/} \longrightarrow B_{v(0)/} \times_B A$ is a left fib. (HTT; 2.1)

As pointwise contractible fibers of $\to$ are contractible. (HTT 2.1) Hence $\to$ is an equiv as it is Kan fibred.

Prop: Let $\mathcal{C}, \mathcal{D}$ be ∞-cats. A map $f : \mathcal{C} \to \mathcal{D}$ is an equivalence iff it is a DK-equiv.

Remark: The cat. of simplicial categories sCat admits a model str. with Weq $\to$ DK equiv, fib. obj are DK-fibrant.

(Bergner Model structure)

Pre-fib sSets

Def: A sSet S is pre-fib. if it's

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a) every map $\Lambda^2_1 \longrightarrow S$ extends along $\Lambda^2_1 \subseteq \Delta^2$

b) for every $0 < i < n$ if $f : \Lambda^n_i \longrightarrow S$ is a map s.t. $d_0 f$ is a constant $(n-1)$ simplex, then f extends to $\Lambda^n_i \subseteq \Delta^n$

Remark: If S is a pre-fib. s-Set then $\forall$ pair of vertices $a, b \in S$ $\mathrm{Hom}^L_S(a,b)$ is a Kan-complex

Proof: S is a pre-fib sSet, $\exists$ an inner anodyne map

$$f : S \longrightarrow R \quad \text{s.t.}$$

(i) R is an ∞-cat.

(ii) the map

$$S_{x/} \times_S \{y\} \longrightarrow R_{x/} \times_R \{y\}$$

is an iso $\forall$ vertices $x, y \in S$

Def: Consider A map $f : S \to R$ of pre-fib sSets is fully faithful if

$$\mathrm{Hom}^L_S(x,y) \longrightarrow \mathrm{Hom}^L_R(f(x), f(y))$$

on the ~~of a~~ naive mapping space is a htpy eqv between Kan-complexes $\forall$ obj x, y of S

Joyal Model Structure

Uniquely det. by sSet fact. that fib. obj. are ∞-cats and cofibs are mono.

Thm $\exists$ a model structure on sSet for which

(i) cofibs are monos

(ii) fib objects are ∞-cats

(iii) w.e. are cat. equivalences.

This model is cofib. generated and left proper.

Proof W of cat. equiv satisfies 2-out-of-3 property.

$C_W = \{$class of monos generated by the set of maps A

and $F = \{$class of maps with rlp wrt maps in $A\}\}$

By the small obj. argument we have a weak factorization system

(C_W, F)

Let $F_W = \{$acyclic fibs$\}$

$C = \{$class of monos in sSet$\}$

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By the small object argument we get a weak factorization system (C, F_W)

$$C_W \subseteq C \cap W \quad \& \quad F_W \subseteq F \cap W.$$

It suffices to prove $\underline{F \cap W \subseteq F_W}$

Suppose $p : X \longrightarrow S$ belongs to $F \cap W$. We can prove that p is an inner fib. such that $\mathrm{ho}(p) : \mathrm{ho}(X) \longrightarrow \mathrm{ho}(S)$ is an isofib. and p is a cat. equiv.

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "p"'] & Y \arrow[d, "q"] \\ S \arrow[r, "v"'] & T \end{tikzcd}$$

u, v are inner anodyne and $q : Y \longrightarrow T$ is an inner fib. & T is an ∞-cat.

It follows that q is cat. equiv

$\Rightarrow q$ is an acyclic Kan fib.

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr, "p"'] & & S \times_T Y \arrow[dl] \\
& S &
\end{tikzcd}$$

Inner Anodyne

Since $p : X \to S$ is an inner f.b. p is a retract of the acyclic Kan f.b. $S \times_T Y \to S$; hence is also an acyclic Kan f.b.

Thanks