

# The Joyal Model Structure

Higher Categories Reading Seminars  
ntpy

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## Plan

- A quick Recap
- Categorical Equivalences
- Dwyer Kan equivalences
- $\text{pre}$ -fibrant simplicial sets
- The Joyal Model Structure

## Recall

We've the functor  $H_0: \text{Set} \rightarrow \text{Cat}$   
 $S \mapsto H_0(S)$

Def: An object  $c: u \rightarrow v$  in an  $\omega$ -category  $\mathcal{C}$  is called an equivalence if the image in  $H_0(\mathcal{C})$  is an iso.

More generally we can replace  $\mathcal{C}$  by an sSet  $S$ .

Def. Let  $\mathcal{C}, \mathcal{D}$  be  $\omega$ -categories. A map  $f: \mathcal{C} \rightarrow \mathcal{D}$  is an equivalence if  $\exists$  a map  $g: \mathcal{D} \rightarrow \mathcal{C}$  such that  $f$  and  $g$  are equivalences  $g \circ f \rightarrow \text{id}_{\mathcal{C}}$  and  $f \circ g \rightarrow \text{id}_{\mathcal{D}}$  in the  $\omega$ -cats  $\text{Fun}(\mathcal{C}, \mathcal{C})$  and  $\text{Fun}(\mathcal{D}, \mathcal{D})$ .

Notation  $f, g: A \rightarrow B$  in sSet are  $J$ -homotopic if  $\exists$  a map  $H: A \times J \rightarrow B$  such that  $H_{i_0} = f$  and  $H_{i_1} = g$  where  $i_0, i_1: A \rightarrow A \times J$ .

$J$  denotes the groupoid interval

(groupoid with objects  $0, 1$  and a unique iso between them)

Remark  $f: \mathcal{C} \rightarrow \mathcal{D}$  is an equivalence of  $\omega$ -categories if it's a  $J$ -hyp equivalence in sense that

$\exists$  a map  $f': \mathcal{D} \rightarrow \mathcal{C}$  such that  $f \circ f'$  and  $f' \circ f$  are  $J$ -homotopic to  $\text{id}_{\mathcal{D}}$  and  $\text{id}_{\mathcal{C}}$ .

Lemma Let  $l: K \rightarrow I$  is a homotopy equivalence,

Lemma Let  $f: \mathcal{K} \rightarrow \mathcal{L}$  is a homotopy equivalence  
 where  $\mathcal{K}, \mathcal{L} \in \underline{\text{Kan}}$ . Then  $f$  is an equivalence of  $\infty$ -cat.

## Categorical Equivalences

Def (Joyal) A map  $f: A \rightarrow B$  of sSet is a cat. equivalence if for  
 any  $\infty$ -cat  $\mathcal{Y}$ , the map

$$\text{Fun}(B, \mathcal{Y}) \rightarrow \text{Fun}(A, \mathcal{Y}) \text{ is an equiv}$$

of  $\infty$ -cats.

Prop: If  $f: \mathcal{C} \rightarrow \mathcal{D}$  is an equivalence between  $\infty$ -cats, then  $f$  is  
 a categorical equivalence.

Lemma If map  $f: A \rightarrow B$  is in this category then  $f$  is a  
 categorical equivalence

(HTT) 2.3. —

Lemma If  $g: X \rightarrow \mathcal{Y}$  is an acyclic fib. then  $g$  is a cat.  
 equivalence

Remark The class of cat. equivalences in sSet is stable under  
 filtered colimits.

Joyal-Kan equivalence (DK-equivalences)

(is an  $\infty$  cat,  $\mathcal{K}, \mathcal{Y} \in \text{obl } \mathcal{L}$ ) then  $\exists$  a mapping space of

morphisms  $\text{Hom}_{\mathcal{L}}(\mathcal{K}, \mathcal{Y})$  (well defined object of  $\text{Lan}(SP)$ )

morphisms  $\text{Hom}_e(n, y)$  (well defined object of  $\text{Ho}(S)$ )

(left morphisms)

Def:  $S \in \text{Set}$  and  $x, y$  are vertices of  $S$ , then the set  $\text{Hom}_S^L(n, y)$  of left morphisms from  $n$  to  $y$  in  $S$  is the set where  $\text{set}$  of  $n$ -simplices in the set of all maps

$$f: \Delta^{n+1} \rightarrow S \text{ with } f(0) = x \quad f|_{\Delta^{n, \dots, n}} \text{ is the}$$

constant  $n$ -simplex on vertex  $y$ .

Lemma If  $\ell$  is an  $\omega$ -cat then  $\text{Hom}_e^L(n, y)$  is a Kan complex presenting  $\text{Map}_e(n, y)$

$S$  present  $\ell \Rightarrow \exists$  a cat. eqn  $S \rightarrow \ell$ .

$$\text{Hom}_S^L(x, y)$$

Def (Dk equivalence?)

$A, B$ :  $\omega$ -cats,  $\gamma: A \rightarrow B$  is

- (i) essentially surjective if  $(\text{Ho } \gamma): \text{Ho}(A) \rightarrow \text{Ho}(B)$  is ess surj.
- (ii) fully faithful if

$$\text{Hom}_A^L(k, k') \xrightarrow{\cong} \text{Hom}_B^L(\gamma(k), \gamma(k'))$$

for any vertices  $k, k' \in A$

$\Rightarrow$  If  $\gamma$  is fully faithful & ess-surj., then  $\gamma$  is a Dwyer-Kan equivalence.

$\dots$   $\omega$ -cats is an eqn if it is a Dk eqn

v

Claim: A map between  $\mathcal{C}$ -sets is an equalizer iff it is a 2M equal

Lemma If  $f: A \rightarrow B$  is an equalizer between  $\mathcal{C}$ -set  $A, B$ , then  $f$  is fully faithful.

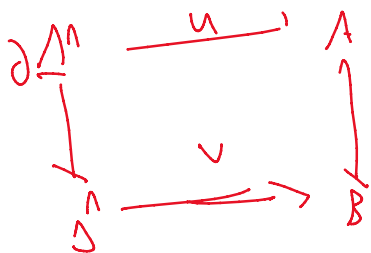
Def: An inner fib. between  $\mathcal{C}$ -sets  $X, Y$   $f: X \rightarrow Y$  is a cat. fib iff  $\text{Ho}(f): \text{Ho}(X) \rightarrow \text{Ho}(Y)$ .

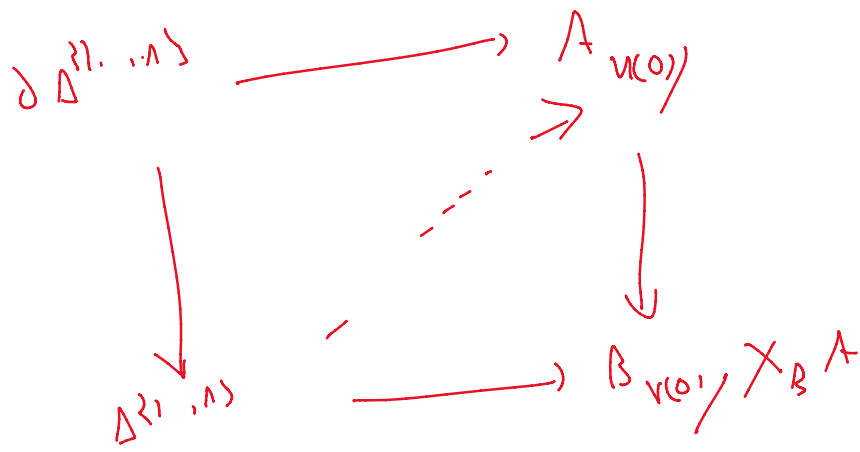
Remark An inner fib.  $f: A \rightarrow B$  is a cat. fib iff it has spt  $\{0\} \subseteq J$

Lemma (Joyal) If  $f: A \rightarrow B$  is a cat. fib. between  $\mathcal{C}$ -set  $A, B$  is fully faithful and essentially surj. then  $f$  is an acyclic Kan. fib.

Proof: Suppose  $f$  is fully faithful & ess. surj.

$f$  has spt  $\{0\} \subseteq J$   
 We prove that for  $n \geq 1$   $f_h$





The induced map

$A_{\text{rel.}} \longrightarrow B_{\text{rel.}} / X_B A$  is a left fib. (HTT, 2.1)  
 As  $\mathcal{D}^{\text{rel.}}$  is a left fibration of  $\mathcal{D}^{\text{rel.}}$  are uncontractible.  
 $\mathcal{D}^{\text{rel.}}$  is an  $\mathcal{D}^{\text{rel.}}$  fibration.

Prop.: Let  $\mathcal{C}, \mathcal{D}$  be  $\mathcal{D}^{\text{rel.}}$ . A map  $f: \mathcal{C} \rightarrow \mathcal{D}$  is an equivalence iff it is a DK-equivalence.

Remark: The cat. of simplicial categories that admit a model structure with  $W_{\text{eq}} \rightarrow \mathcal{D}^{\text{rel.}}$  fib. obj are  $\mathcal{D}^{\text{rel.}}$ -fibrant.

(Bergner Model structure)

Pre-fib Sfts

Def.: A sft  $\mathcal{S}$  is pre-fib. if its

Def. A set  $S$  is pre-fib. if its

- a) every map  $\Delta^2 \rightarrow S$  extends to  $\Delta^2 \subseteq \Delta^2$
- b) for every  $0 < i < n$  if  $f: \Delta_i^1 \rightarrow S$  is a map  $S \rightarrow S$  of  $S$  is a constant  $(\rightarrow)$  simplex, then  $f$  extends to  $\Delta_i^1 \subseteq \Delta^1$

Thm. If  $S$  is a pre-fib. set then  $\forall$  pair of vertices  $a, b \in S$   $\text{Hom}_S^L(a, b)$  is a Kan-complex

Proof:  $S$  is a pre-fib. set,  $\exists$  an inner anodyne map

$$f: S \rightarrow R \text{ st.}$$

- (i)  $R$  is an  $\omega$ -cat.
- (ii) the map

$$\text{Hom}_S^L(x, y) \longrightarrow \text{Hom}_R^L(x, y)$$

is an iso  $\forall$  vertices  $x, y \in S$

Conseq.  $\forall$  map  $f: S \rightarrow R$  of pre-fib. sets is fully faithful

$$\text{Hom}_S^L(x, y) \xrightarrow{\cong} \text{Hom}_R^L(f(x), f(y)) \text{ in } \text{Ho} S$$

$\Rightarrow$  a naive mapping space is a fully eqn between Kan-complexes  $\forall$  obj  $x, y \in S$

Tyrd Model S-Structure

Uniquely det. by <sup>the</sup> fact that lib. obj. are  $\omega$ -cets and  
 obj's are num.

Thm  $\exists$  a Model structure on  $S$  for which

- (i) obj's are numos
- (ii) lib objects are  $\omega$ -cets
- (iii) wqf are cat. equiv.

This model is wqf-generated and left proper.

Proof (  $\mathcal{W}$  of cat. eqval  $\text{subcat}$ : 2-out-3  $\rightarrow$  prop. )

$\mathcal{C}_w = \{ \text{class of numer generated by the set of maps } A \}$

and  $\mathcal{F} = \{ \text{class of maps into obj with maps in } \mathcal{A} \}$

By the small obj. argum we've a weak factor system

$(\mathcal{W}, \mathcal{F})$

let  $\mathcal{F}_w = \{ \text{any class with lib} \}$

$\mathcal{C} = \{ \text{class of numer with } \mathcal{F} \}$



$C = \{ \text{class of numbers w/ rset} \}$

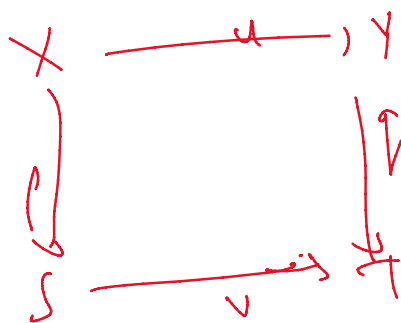
By the small object argument we get a weak factorization sysd.  $(C, F_w)$

$$C_w \subseteq C \cap W \quad \Delta \quad F_w \subseteq F \cap W.$$

It suffices to prove  $F \cap W \subseteq F_w$

Suppose  $p: X \rightarrow S$  belongs to  $F \cap W$   
 we can prove that  $p$  is an inner fib. such that

$(Ho(p): Ho(X) \rightarrow Ho(S))$  is an isofib. and  $p$  is a cat. eqv.



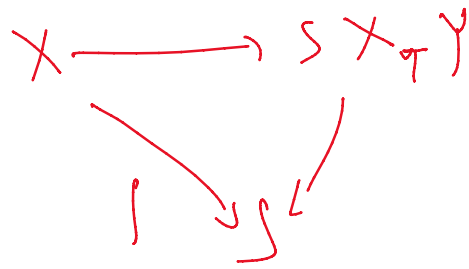
$u, v$  are inner anodyne

and  $r: Y \rightarrow T$  is an inner fib.

$\hookrightarrow T$  is an  $\mathcal{A}$ -cat.

It follows that  $q$  is cat. eqv

$\Rightarrow q$  is an anodyne Kan fib.



Innen Anzug

Since  $p: X \rightarrow S$  is an inner v.h. cyclic Van v.b.  $p$  is a retract of  $h_1$  cyclic Van v.b.

$$S \times_T Y \xrightarrow{\quad} S \quad \hookrightarrow$$

Hence it is also an cyclic Van v.b.

Thanks