$C \to NhC$

$(\alpha: \Delta^n \to C) \mapsto (\alpha(0) \to \alpha(1) \to \cdots \to \alpha(n))$ in hC

$(f: \Delta^1 \to C) \mapsto (x \xrightarrow{[f]} y)$ in hC.

If $[f]$ an iso... $\exists\ [f^{-1}]: y \to x$ in hC.

i.e. $x \xrightarrow{[f]} y \xrightarrow{[f]^{-1}} x$, $x \xrightarrow{[id]} x$ commutes.

Call this $\alpha: [2] \to hC$

$\rightsquigarrow$

$N\alpha: \Delta^2 \to NhC$

$$\begin{tikzcd} \Lambda^2_0 \arrow[r, "\binom{f}{id}"] \arrow[d] & C \arrow[d] \\ \Delta^2 \arrow[r, "N\alpha"] & NhC \end{tikzcd}$$

$C \xrightarrow{\varphi} NhC$ is cons. iff $hC \to hNhC$ is...

$\cong$

hC id?

$hC \xrightarrow{h\varphi} hNhC$

$x \longmapsto x$

$[f] \downarrow\ \longmapsto\ \downarrow h\varphi[f] = [\varphi f] \circ [f]$.

$y \qquad y$

$$\begin{tikzcd} & \cdot \arrow[dr, "\beta"] & \\ C \arrow[ur, "\alpha"] \arrow[rr, "id"'] & & C \end{tikzcd}$$

Say β, $\beta \circ \alpha$ are cons.

$\Rightarrow \alpha$ cons.

?

$f: \Delta^1 \to C$

---

basepoints. Set_* & Top_*.

sets: $\ni S = \{s_1, s_2, \ldots, s_n\}$.

(pick one element).

$Set_* \ni S = \{s_1, s_2, \ldots, s_n\}$.

basepoint.

$Set_* \subseteq Set$

" {sets w/ 1 pt.}.

$Hom(Set_*)$

(X, x_0)

$\downarrow f$ is a set-function

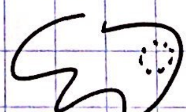
$f: X \to Y$

$x_0 \mapsto f(x_0) = y_0$.

(Y, y_0)

$X = \{x_1, \ldots, x_n\}$

$Y = \{y_0, \ldots, y_m\}$.



$(- \circ f)$

$X \xrightarrow{f} Y$, $\alpha: Y \to Z$, $X \xrightarrow{\alpha \circ f} Z$

pull back — $f^*\alpha$ "

$Z \xrightarrow{f_*\alpha} Y$, $\alpha \downarrow$, $X \xrightarrow{f} Y$ — push forward.

$(f \circ -)$

Set_* = ob: sets w/ maps

$\{*\} \xrightarrow{x_0} X$

singleton

Top_*: ob: "ptd" top-spaces.

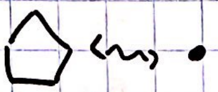
X a space comes w/

$* \xrightarrow{x_0} X$

G: a group. $\{g, g^{-1}, \dots\}$. elements. e

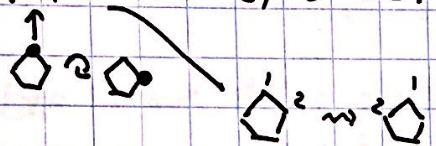
BG — ob: $\bullet$ $\circlearrowleft$ "id" e — ~~an elt in G?~~

"ie" — hom: $\{\bullet \xrightarrow{g} \bullet\}_{g \in G}$. /maps.

D_5 :: symmetries of  $\bullet$

$\cong$ D_{10} ''

$\{\sigma, \tau : \sigma^5 = e,\ \tau^2 = e\}$.



composition in BG:

$$\begin{tikzcd} & \bullet \arrow[dr, "h"] & \\ \bullet \arrow[ur, "g"] \arrow[rr, "h\circ g"'] & & \bullet \end{tikzcd}$$

$h \circ g = hg$

ex//. Given a gp. hom.

$G \xrightarrow{\varphi} H$

$g \mapsto \varphi(g)$

$BG \dashrightarrow BH$, $B\varphi$

functor

$$\begin{tikzcd} \bullet_G \arrow[r, mapsto] \arrow[d, "g"] \arrow[dd, bend right, "g' \circ g"'] & \bullet_H \arrow[d, "\varphi(g)"] \arrow[dd, bend left, "\varphi(g')\circ\varphi(g) = \varphi(g \circ g')"] \\ \bullet_G \arrow[d, "g'"] & \bullet_H \arrow[d, "\varphi(g')"] \\ \bullet & \bullet \end{tikzcd}$$

(i) check $B\varphi$ is a functor

- $id \mapsto id$
- comp $\mapsto$ comp.

(ii) a gp hom. has some properties.

- $e_G \mapsto e_H$
- $g^{-1} \mapsto \varphi(g)^{-1}$

⋮ check how these look on the functor

locally small: for any objects $X, Y \in \mathcal{C}$

$\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is a set.

bijection is an iso. in Sets.

small: Take all morphisms in $\mathcal{C}$

$$\mathrm{Hom}(\mathcal{C}) := \bigcup_{X,Y \in ob(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(X,Y)$$

If this guy is a set, then $\mathcal{C}$ is "small".

Groups is · locally small?

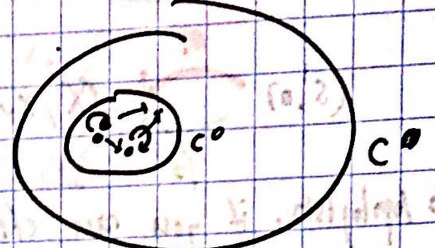
$\mathrm{Hom}_{Gps}(G,H)$ is a set. ✓.

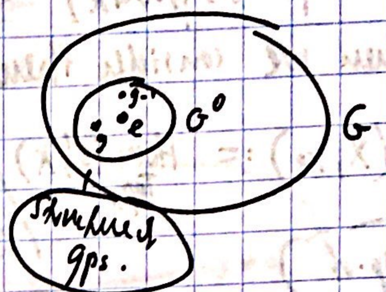
· but not small? —

i.e. $\bigcup_{G,H} \mathrm{Hom}_{Gp}(G,H)$.

subcat.

A subgp. should have the structure of a gp. & live inside of a larger gp.





A subcat. should have the structure of a cat (1) & live inside a larger cat (2). $\mathcal{C}^0 \subseteq \mathcal{C}$

(2) $\cdot\, ob(\mathcal{C}^0) \subseteq ob(\mathcal{C})$
$\cdot\, mor(\mathcal{C}^0) \subseteq mor(\mathcal{C})$

(1) (structure of a cat.)
· every object has an id $\rightsquigarrow$ $\mathcal{C}^0$ to contain all id's of ob in $\mathcal{C}^0$.
· composition is well-defined $\rightsquigarrow$ $\mathcal{C}^0$ should contain compositions of all its maps.
& they should be the same as in $\mathcal{C}$.

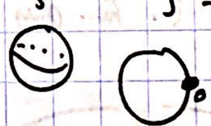
eg. when showing $\mathcal{C}^{mono.}$ – ob = ob($\mathcal{C}$)
mor = { mono's in $\mathcal{C}$ }.

to show $\mathcal{C}^{mono.} \underset{subcat.}{\subseteq} \mathcal{C}$

WTS: (2) ✓
(1) · identities are in $\mathcal{C}^{mono}$ (i.e. every id is a mono.)
· comp. of mono's are mono. (ex).

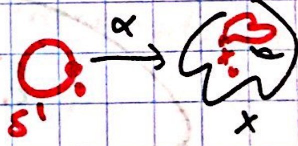
(ex?) iso's $\overset{?}{\Rightarrow}$ mono's.

Take $(X, x_0) \in Top_*$.
Take $(S^1, 0) \in$
circle.



$S^1 = \{|z| = 1\} \subseteq \mathbb{C}$

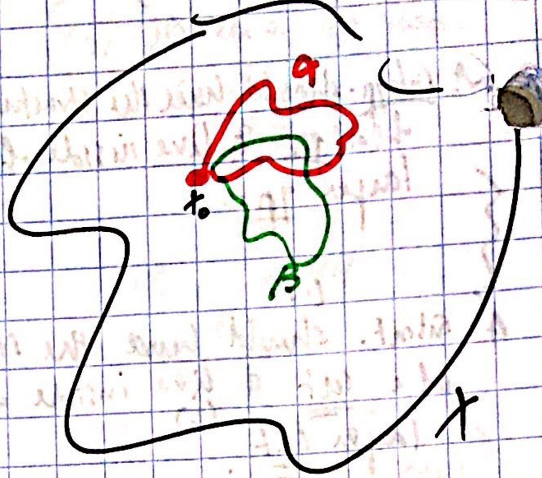
Look at $Hom_{Top_*}((S^1,0),(X,x_0)) = \{$ ptd cts maps $S^1 \to X$, $0 \mapsto x_0$ $\}$.



forms a set... but turns out you can give it a gp. structure.

So an elt. ~~$\pi_1(X, x_0)$~~ is $\alpha: (S^1, 0) \to (X, x_0)$ is a "loop" in X.

$$(S^1, 0) \underset{\beta}{\overset{\alpha}{\rightrightarrows}} (X, x_0)$$



To topologists, if you can cts'ly deform "homotopy"
$\alpha \rightsquigarrow \beta$ (or $\beta \rightsquigarrow \alpha$)
then we consider them the same.

$$\pi_1(X, x_0) := \mathrm{Hom}_{\mathrm{Top}_*}(S^1, X) \Big/ \underset{\text{homotopy}}{\sim}$$

Fund. gp. of a top. space. ptd! (X, x_0)

$= \{$ equiv. classes $[\alpha]$, where $[\alpha] = [\beta]$ iff $\alpha \underset{\text{htpic}}{\sim} \beta \}$.

(another notation).

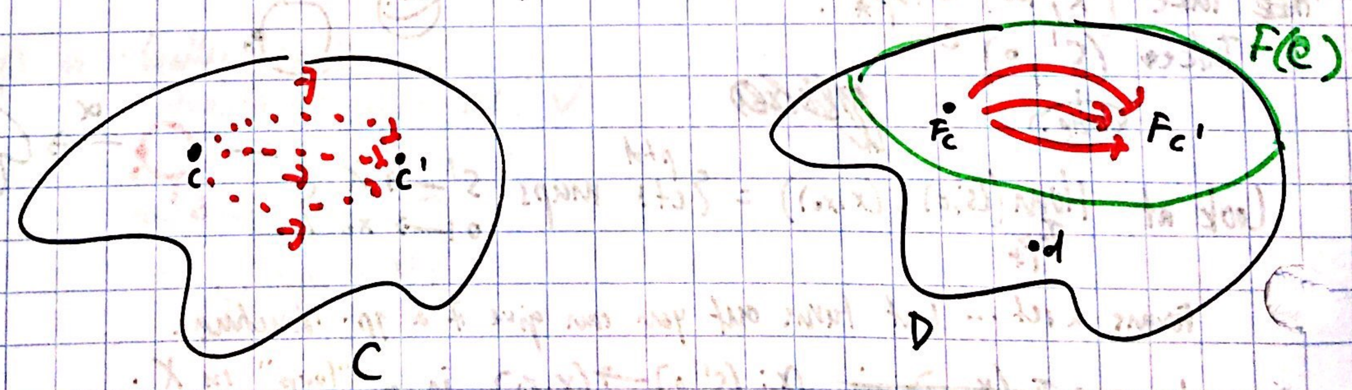
$\pi_1(X, x_0) = [S^1, X]_*$ — ptd maps.

sq. brackets = htpy classes.

Q:
When are two categories $\mathcal{C}$ & $\mathcal{D}$ "the same"?
i.e. what is an isomorphism in <u>Cat</u>? $\rightsquigarrow$ A: a functor that is fully faithful & ess. surjective.

(i) Firstly a functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$
is a function $\mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D})$ — ess. surj.
. functions $\mathrm{Hom}_{\mathcal{C}}(c, c') \to \mathrm{Hom}_{\mathcal{D}}(Fc, Fc')$ — want to look at for fully faithful.
satisfying some things..

(i) F is $\boxed{\text{full}}$ if each function $\mathrm{Hom}_{\mathcal{C}}(c, c') \to \mathrm{Hom}_{\mathcal{D}}(Fc, Fc')$
is <u>surjective</u>.
i.e. for any map $Fc \to Fc'$ in $\mathcal{D}$,



(ii) <u>faithful</u> if $\mathrm{Hom}_{\mathcal{C}}(c, c') \to \mathrm{Hom}_{\mathcal{D}}(Fc \to Fc')$ is inj.

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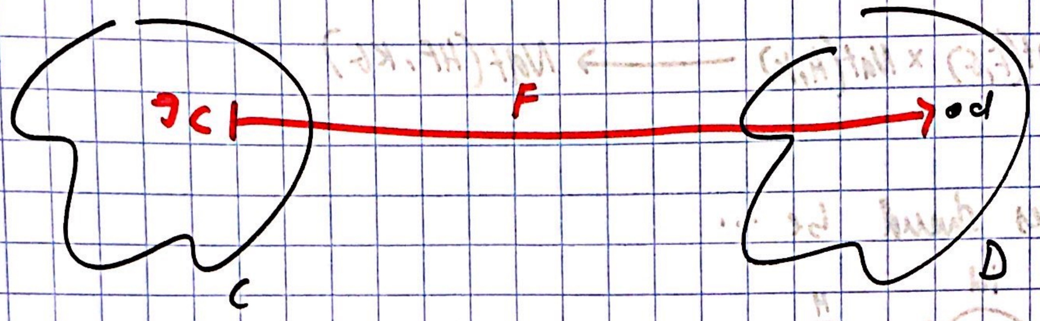
fully-faithful: $\mathrm{Hom}_C(c,c') \xrightarrow{\sim} \mathrm{Hom}_D(Fc,Fc')$

are bijections of sets.

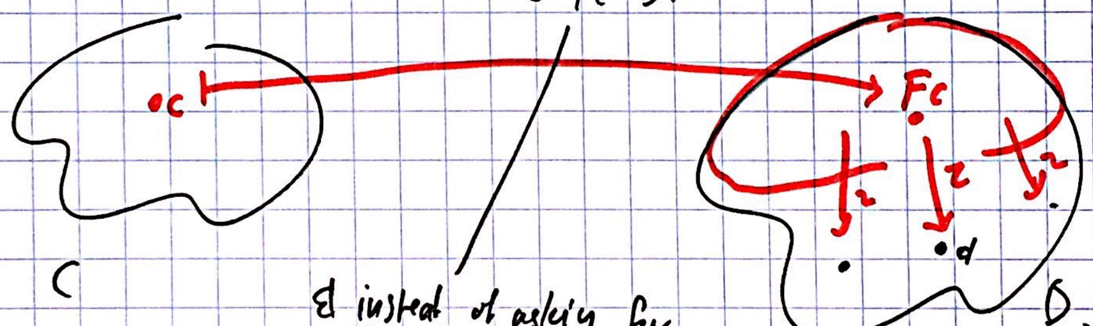
essentially surjective is a condition on objects.

Guess: For any $d \in D$, there should be a $c \in C$ w/ $Fc = d$



But this is too strong.

We take instead a slightly weaker version...
For any $d \in D$, there's a $c \in C$ s.t. $Fc \in D$.



& instead of asking for equality $Fc = d$
we ask instead for an iso. $Fc \xrightarrow{\sim} d$

injectivity on objects isn't required.

---

vertical

F

C: same cat.

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & y \arrow[r, "g"] & z \end{tikzcd}$$

composition in ordinary cats. is a function (set-morphism?)

$$\mathrm{Hom}_C(x,y) \times \mathrm{Hom}_C(y,z) \xrightarrow{-\circ-} \mathrm{Hom}_C(x,z)$$

cartesian product.

$$(f, g) \longmapsto g\circ f$$

$A, B \in \mathrm{Set}$. $\mathbb{R}^2 = \mathbb{R}\times\mathbb{R}$.
$A \times B = \{(a,b) : a \in A, b \in B\}$.

in 2-cats horizontal comp.

$$\begin{tikzcd} C \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & D \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\Downarrow\beta"] & E \end{tikzcd}$$

$(\underset{:F \to G}{\alpha}, \underset{:H \to K}{\beta}) \rightsquigarrow (\beta * \alpha)$

$Nat(F,G) \times Nat(H,K) \longrightarrow Nat(HF, KG)$

identities should be ...

$$\begin{tikzcd} C \arrow[r, bend left, "id"] \arrow[r, bend right, "id"'] \arrow[r, phantom, "\Downarrow id"] & C \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\beta\Downarrow"] & D \end{tikzcd}$$

$(\beta * id) = \beta.$

$H \circ id \to K \circ id$

$H \to K.$