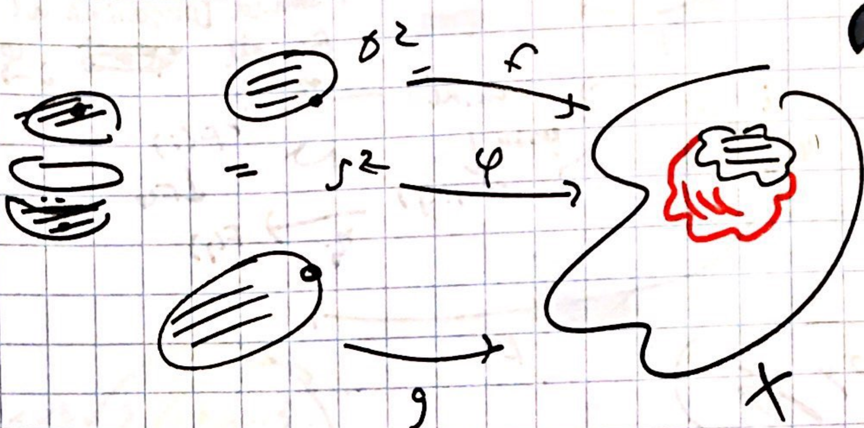
$$\begin{tikzcd}
S^1 \arrow[r] \arrow[d] & D^2 \arrow[d] \arrow[ddr, bend left] & \\
D^2 \arrow[r] \arrow[drr, bend right] & S^2 \arrow[dr, dotted] & \\
 & & X
\end{tikzcd}$$



4/20/22.

Say C, D are categories.

Look @ a category called $\mathrm{Fun}(C,D)$ — ob = $\{\text{functors } C \xrightarrow{F} D\}$.
mor = $\{\text{nat'l transformations } \alpha: F \Rightarrow G\}$.

ex/ check this is a cat
- identities $\forall F \in \mathrm{Fun}(C,D)$
  i.e. a nat'l transf. $id_F : F \Rightarrow F$
- composition, $F \overset{\alpha}{\Rightarrow} G \overset{\beta}{\Rightarrow} H$, $\beta \circ \alpha$

Now... in any cat, we have the notion of an isomorphism.

($f \in \mathcal{C}$ is an iso.) iff ($\exists f^{-1} \in \mathcal{C}$ s.t.

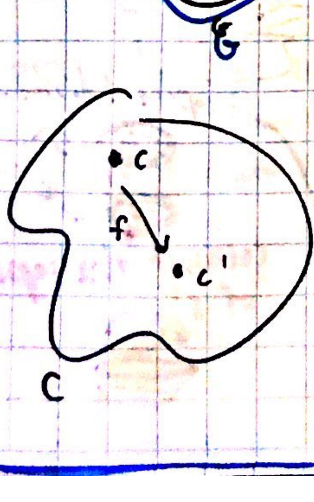
$$\begin{tikzcd}
 & \bullet \arrow[dr, "f^{-1}"] & \\
\bullet \arrow[ur, "f"] \arrow[rr, "id"'] & & \bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \bullet \arrow[dr, "f"] & \\
\bullet \arrow[ur, "f^{-1}"] \arrow[rr, "id"'] & & \bullet
\end{tikzcd}$$
)

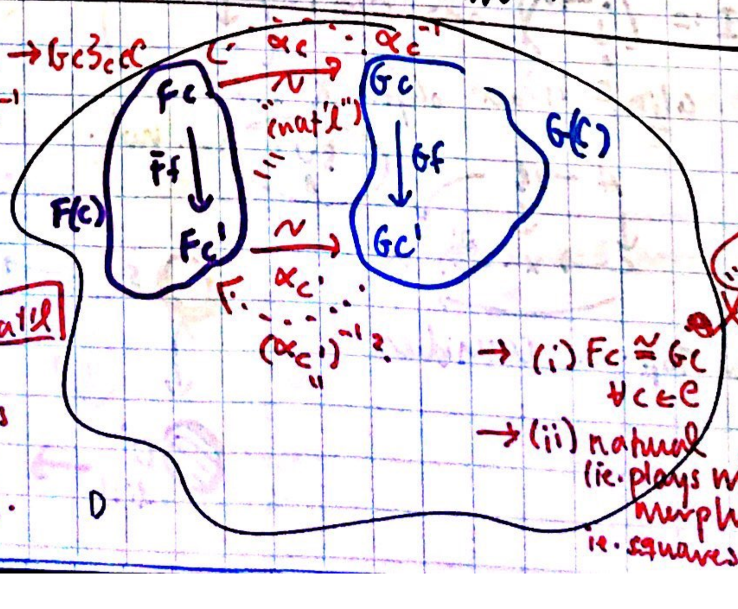
eg. in $\mathcal{C}$ = Sets, isos are bijections.
in $\mathcal{C}$ = Groups, isos are gp. isomorphisms.
in $\mathcal{C} = \mathrm{Fun}(C,D)$, isos are nat'l isomorphisms.
i.e. they tell you when two functors are "the same".

Say $\begin{tikzcd} C \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & D \end{tikzcd}$ $\alpha = \{\alpha_c : Fc \to Gc\}_{c \in C}$

what should α^{-1} be?
$\alpha^{-1} : G \Rightarrow F$
$\{Gc \xrightarrow{(\alpha^{-1})_c} Fc\}$

Def α is a $\boxed{\text{nat'l}}$ $\boxed{\text{iso}}$ iff all the maps $\alpha_c : Fc \to Gc$ are invertible.





→ (i) $Fc \cong Gc$ $\forall c \in \mathcal{C}$ "pointwise bijective"
→ (ii) natural (ie. plays nicely w/ morphisms. ie. squares commute)

ex/1. Show that ($0 \in \mathcal{C}$ is initial) $\Leftrightarrow$ (there's a nat'l iso.)

$$\Big(\exists!\text{ map } 0 \to x \ \forall x \in \mathcal{C}\Big) \Leftrightarrow \Big(\mathrm{Hom}_{\mathcal{C}}(0,x) \cong \{*\} \ \forall x \in \mathcal{C}\Big) \Leftrightarrow \Big(\mathcal{C} \overset{\mathrm{Hom}_{\mathcal{C}}(0,-)}{\underset{*}{\rightrightarrows}} \text{Sets}, \ \Downarrow\!\sim\Big)$$

$* =$ const. $*$

$$\begin{aligned} * : \mathcal{C} &\to \text{Set} \\ c &\mapsto \{*\} \\ f\downarrow \quad & \quad \downarrow ! = id_{\{*\}} \\ c' &\mapsto \{*\} \end{aligned}$$

<u>Yoneda</u>: (z.z.u)

Say $F: \mathcal{C} \to \text{Set}$.

$\mathcal{C}$ is locally small (i.e. $\mathrm{Hom}(x,y)$ are sets).

$x \in \mathcal{C}$ an object.

There's a bijection (of <u>sets</u>)

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{Set})}(\mathrm{Hom}_{\mathcal{C}}(x,-), F) \cong Fx$$

$\{$nat'l transfo's $\mathrm{Hom}_{\mathcal{C}}(x,-) \Rightarrow F\}$

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(x,-) : \mathcal{C} &\to \text{Set} \\ y &\mapsto \mathrm{Hom}_{\mathcal{C}}(x,y) \\ \varphi \downarrow \quad & \quad \downarrow \varphi\circ - \\ z &\mapsto \mathrm{Hom}_{\mathcal{C}}(x,z) \end{aligned}$$

i.e. $\mathrm{Hom}_{\mathcal{C}}(x,-)$ is an <u>object</u> in the functor cat $\mathrm{Fun}(\mathcal{C},\text{Set})$

$$\alpha \longmapsto \alpha_x(id_x)$$

$id_x \in \mathrm{Hom}_{\mathcal{C}}(x,x)$

$\{\alpha_c : \mathrm{Hom}_{\mathcal{C}}(x,c) \to Fc\}_{c \in \mathcal{C}}$

in particular, take $c = x$

$\alpha_x : \mathrm{Hom}_{\mathcal{C}}(x,x) \to Fx$

$id_x \longmapsto \alpha_x(id_x)$

We want an inverse ... i.e. a map $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{Set})}(\mathrm{Hom}_{\mathcal{C}}(c,-), F) \xleftarrow{\Psi} Fc$

$\Psi(x) \longleftarrow\!\shortmid x$

so... given $x \in Fc$

we want a nat'l transf. $\Psi(x) : \mathrm{Hom}_{\mathcal{C}}(c,-) \Rightarrow F$

i.e. $\{\Psi(x)_d : \mathrm{Hom}_{\mathcal{C}}(c,d) \to Fd\}_{d \in \mathcal{C}}$

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(c,d) & \to & Fd \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{C}}(c,d') & \to & Fd' \end{array}$$

In particular for $d = c$, $d' = e$

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(c,c) & \to & Fc \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{C}}(c,e) & \to & Fe \end{array}$$

this is the 1st square on pp. 57.

check this is an inverse to $\Phi : \mathrm{Hom}(\mathcal{C}(c,-), F) \to Fc$

i.e. want

What should $\Psi(x)_d : \mathrm{Hom}_{\mathcal{C}}(c,d) \longrightarrow Fd$ be?

We know that this commutes: for $(c \xrightarrow{f} d)$

$$\begin{tikzcd}
\mathcal{C}(c,c) \arrow[r, "\Psi(x)_c"] \arrow[d, "f\circ -"'] & Fc \arrow[d, "Ff"] \\
\mathcal{C}(c,d) \arrow[r, "\Psi(x)_d"'] & Fd
\end{tikzcd}$$

Take $id_c \longmapsto \Psi(x)_c(id_c)$

$$\begin{tikzcd}
id_c \arrow[r, mapsto] \arrow[d, mapsto] & \Psi(x)_c(id_c) \arrow[d, mapsto] \\
f\circ id_c = f \arrow[r, mapsto] & \Psi(x)_d(f) = Ff\left(\Psi(x)_c(id_c)\right)
\end{tikzcd}$$

$\Psi(x)_d(f) = Ff\left(\Psi(x)_c(id_c)\right) = \; ?$

We know also that Ψ should be Φ^{-1}

Fun(C,Set):

$\mathrm{Hom}_{\mathcal{C}}(c,c) \xrightarrow{\Psi\,\Phi} Fc \xrightarrow{\Psi = \Phi^{-1}\;\Phi} \mathrm{Hom}_{\mathcal{C}}(c,c)$

$id_c \xrightarrow{\;id\;} id_c$

$\Phi(x)_c$

$x \in Fc$

$id_c \xrightarrow{\Phi} \Phi(id_c) \xrightarrow{\Psi} \Psi(\Phi id_c) = id_c$

$id_c \longmapsto id_c$

$Fc \xrightarrow{\Psi} \mathrm{Hom}_{\mathcal{C}}(c,c) \xrightarrow{\Phi} Fc$ (since $\Psi = \Phi^{-1}$)

$x \xrightarrow{\Psi} \Psi(x)$, $\quad x \longmapsto x$

$$\begin{tikzcd}
\mathcal{C}(c,c) \arrow[r] & Fc \\
id_c \arrow[r, mapsto] & \Psi(x)_c(id_c)
\end{tikzcd}$$

$\Psi(x)_c(id_c) \;\|\text{(claim)}\; x$

$$\begin{tikzcd}
& \mathrm{Nat}(\mathcal{C}(c,-),F) \arrow[dr, "\Phi"] & \\
Fc \arrow[ur, "\Psi"] \arrow[rr] & & Fc
\end{tikzcd}$$

~~Fun(C,Set)~~

this is the same as $\mathrm{Hom}(\mathcal{C}(c,-),F)$ Fun(C,Set)

$x \mapsto \Psi(x) \xrightarrow{\;id\;\Phi\;} \Phi(\Psi(x)) = \Psi(x)_c(id_c)$ by definition of Φ

$x \longmapsto x$