What the heck is a...
universal property?.

still vague but "concrete" Δ: it's a statement about hom-sets.
(some isomorphism)

Roughly... Yoneda's lemma says that in a cat. $\mathcal{C}$
an object $x \in ob(\mathcal{C})$ is determined by $\begin{cases} \bullet \text{ maps into } x \\ \cdot \text{ maps out of } x. \end{cases}$

ie... if you want to study $x \in \mathcal{C}$
you can look @... $\begin{cases} \cdot \bigcup_{y \in \mathcal{C}} Hom_{\mathcal{C}}(y,x) \\ \cdot \bigcup_{y \in \mathcal{C}} Hom_{\mathcal{C}}(x,y) \end{cases}$

more formally... representable functors are those of the form. (ie. naturally isomorphic to.)

$$
\begin{tikzcd}
Hom_{\mathcal{C}}(x,-) : \mathcal{C} \arrow[r] & Sets. & \\
y \arrow[r, mapsto] \arrow[d, "f"'] & Hom_{\mathcal{C}}(x,y) \ni \arrow[d, "f\circ-"] & (x \xrightarrow{\alpha} y) \arrow[d, mapsto] \\
y' \arrow[r, mapsto] & Hom_{\mathcal{C}}(x,y') & (x \xrightarrow{\alpha} y \xrightarrow{f} y')
\end{tikzcd}
$$

There's a Yoneda embedding.. $\{x \mapsto Hom_{\mathcal{C}}(-,x)\}$

Then a universal property (a statement about hom-sets) can determine a limit/colimit (up to isomorphism)

eg. products in Sets. Take sets X, Y...

$$
\begin{tikzcd}
& P := X \times Y \arrow[dl] \arrow[dr] & \\
X & & Y
\end{tikzcd}
$$

U.P.: The product P of X & Y has the property...

$$Hom_{set}(W,P) \cong Hom_{set}(W,X) \times Hom_{set}(W,Y).$$

ie. given a pair of maps $\begin{pmatrix} W \to X \\ W \to Y \end{pmatrix}$, we can find $\exists$ a unique $!$ map $(W \to P)$

s.t.
$$
\begin{tikzcd}
W \arrow[r, dashed, "\exists!"] \arrow[d] \arrow[dr] & P \arrow[d] \arrow[dl] \\
X & Y
\end{tikzcd}
$$
commutes..

exJ. Show that the cart. product $P := X \times Y$ satisfies this U.P.

$\exists$ X, Y … what's the "closest thing" a mapping out of both X, Y.

"unique (up to iso)"

$$\begin{tikzcd} X \arrow[dr] \arrow[drr, "\alpha"] & Y \arrow[d] \arrow[dr, "\beta"] & \\ & C \arrow[r, dotted, "\exists!"] & W \end{tikzcd}$$

C=

X, Y, ….

(ex.) product in a poset-category

$$\begin{tikzcd} & P & \\ X \arrow[ur, phantom, "\geq"] & & Y \arrow[ul, phantom, "\leq"] \end{tikzcd}$$

$C = \{\text{all } x \in X\} \cup \{\text{all } y \in Y\} \cup \{\text{extra..}\}.$

$W.$ $\quad x \mapsto \alpha x$ $\quad y \mapsto \beta y$ $\quad \mapsto w_? \ldots w'$

so the bare minimum requires just $\{x \in X\} \cup \{y \in Y\}$.
ie. the coproduct

$C = X \sqcup Y.$

(?) Think how is the product the "bare minimum" in terms its u.p.

~~A limit~~ Say I is an "indexing cat." & a functor $I \xrightarrow{F} C$
$\quad = \{\circ \to \circ\}.$ picks out a diagram $(X \xrightarrow{f} Y)$

Then one can talk about $\lim\limits_{I} F$ — an object L in C + maps $L \to X$, $L \to Y$.
w/ the u.p… given a W w/ maps

$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr] & L \arrow[d] \arrow[dr, phantom, "="] \arrow[dr] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

$$\begin{tikzcd} W \arrow[d] \arrow[dr] \arrow[d, phantom, shift left=2, "="] & \\ X \arrow[r] & Y \end{tikzcd}$$

then $\exists!$ $W \to L$. making everything commute.

• Can write a u.p. in terms of hom-sets.

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constant diagram..

$$\begin{tikzcd} & \bullet c \arrow[d, "id"] \\ \bullet c \arrow[r, "id"] & \bullet c \end{tikzcd}$$

I

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

$$\begin{tikzcd} & & & c \arrow[d, "id"] \arrow[dd, bend left] \\ I \arrow[rr, bend left, "const."] \arrow[rr, phantom, "\eta \Downarrow"] \arrow[rr, "F"'] & & c \arrow[r, "id"] & c \\ & & & x \arrow[d, "f"] \\ & & y \arrow[r, "g"] & z \end{tikzcd}$$

a nat'l transf. $\eta: \alpha \to \beta$.
η_c :

$$\begin{tikzcd} I \arrow[r, bend left, "\mathrm{const}.c"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{C} \end{tikzcd}$$

$\eta - \{\eta_i : \mathrm{const}(i) \longrightarrow F(i)\}_{i \in I}$.
a map in $\mathcal{C}$.

. should compatible w/ comp., id's, etc...
for all $i \xrightarrow{\varphi} j \in \mathrm{Mor}(I)$

$$\begin{tikzcd} i \arrow[d, "\varphi"'] & \mathrm{const}(i) \arrow[r, "\eta_i"] \arrow[d, "\mathrm{const}\varphi"'] \arrow[dr, phantom, "\wr"] & F(i) \arrow[d, "F\varphi"] \\ j & \mathrm{const}(j) \arrow[r, "\eta_j"'] & F(j) \end{tikzcd}$$

F

ex/

$$\begin{tikzcd} c \arrow[r] \arrow[d, equal] & F_i \arrow[d] \\ c \arrow[r] & F_j \end{tikzcd} \rightsquigarrow \begin{tikzcd} c \arrow[r] \arrow[dr] & F_i \arrow[d] \\ & F_j \end{tikzcd}$$

Show (that a nat'l transfo. $\eta : \mathrm{const}(c) \longrightarrow F$) is the same as (a cone over F.)

Do the case $I = \left(\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \right)$ $\exists!$

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ex/ Intuitively, the terminal object $t \in \mathcal{C}$ is s.t. $\forall\ x \in \mathcal{C}$
$\exists!\ x \xrightarrow{\exists!} t$
i.e. "t sits at the end of $\mathcal{C}$ ... kind of".

$$\begin{tikzcd} w \arrow[dr, dotted, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & L = X \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d] \\ & X \arrow[r] & Z \end{tikzcd}$$

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e.g. Look @ pushout squares.
i.e. colimits of a diagram
$I := \left(\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet \end{tikzcd} \right) \rightsquigarrow F(I) = \left(\begin{tikzcd} A \arrow[r] \arrow[d] & B \\ C \end{tikzcd} \right)$

& $\mathrm{colim}_I F$ is an object X s.t. $\quad B \cup_A C$ — not.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \arrow[dr, bend left] & \\ C \arrow[r] \arrow[rr, bend right] & X \arrow[r, dotted, "\exists!"] & w \end{tikzcd}$$

$X \ldots \exists! \ldots$ t, s

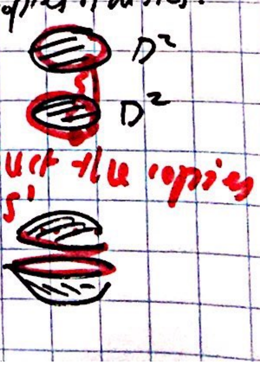
in Top ... consider

$$\begin{tikzcd} S^1 \arrow[r, hook] \arrow[d, hook] & D^2 \arrow[d] \\ D^2 \arrow[r] & D^2 \cup_{S^1} D^2 \end{tikzcd}$$

disk

$D^2 \cup_{S^1} D^2 = S^2$

2 copies of disks.



$$\begin{tikzcd}
S^1 \arrow[r] \arrow[d] & D^2 \arrow[d] \arrow[ddr, bend left] & \\
D^2 \arrow[r] \arrow[drr, bend right] & S^2 \arrow[dr, dotted] & \\
& & X
\end{tikzcd}$$