$$\mathbb{Z}(-) : \text{Groups} \rightleftarrows \text{Rings} : (-)^\times$$

$$\begin{tikzcd}
G \arrow[r, mapsto] \arrow[d, "f"'] & \mathbb{Z}G = \{\sum_{g\in G} \alpha_g \cdot g\} \arrow[d, "\mathbb{Z}f"] \arrow[r, phantom, "\ni"] & (\overset{\alpha_g}{3}g + \overset{\alpha_{g'}}{4}g' + \dots) \arrow[d, mapsto] \\
H & \mathbb{Z}H = \sum_{f(g)\in H} \alpha_g \cdot f(g) & (\underset{\alpha_g}{3}f(g) + \underset{\alpha_{g'}}{4}f(g') + \dots)
\end{tikzcd}$$

$$\begin{tikzcd}
R^\times \arrow[d] & R \arrow[l, mapsto] \arrow[d, "\varphi"] \arrow[r, phantom, "\supseteq"] & R^\times \arrow[d, mapsto] \arrow[r, dash] & \boxed{\text{check!}} \\
S^\times & S \arrow[l, mapsto] & \varphi(R^\times) \\
& \circlearrowleft & S^\times \arrow[u, phantom, "\supseteq"]
\end{tikzcd}$$

Claim: $\mathbb{Z}(-) \dashv (-)^\times$ is left adjoint to.

(i) bijections $\varphi : \mathrm{Hom}_{\text{Rings}}(\mathbb{Z}G, R) \xrightarrow{\sim} \mathrm{Hom}_{\text{Groups}}(G, R^\times)$?

$\forall G \in Gr, R \in Ring'$.

(ii) natural : for morphisms

$$\begin{tikzcd}
G \arrow[r, "f"] & H \\
R \arrow[r, "g"] & S
\end{tikzcd}$$

commuting squares. (in Sets)

$$\begin{tikzcd}
\mathrm{Hom}_{Ri}(\mathbb{Z}G, R) \arrow[r, "\sim"] \arrow[d] & \mathrm{Hom}_{Gp}(G, R^\times) \arrow[d] \\
\mathrm{Hom}_{Ri}(\mathbb{Z}H, S) \arrow[r, "\sim"'] & \mathrm{Hom}_{Gp}(H, S^\times)
\end{tikzcd}$$

(nat'lity in Rings...)
ie. for a ring hom. $R \xrightarrow{f} R'$
the square

$$\begin{tikzcd}
Ri(\mathbb{Z}G, R) \arrow[r, "\sim"] \arrow[d, "f\circ -"'] & Gps(G, R^\times) \arrow[d, "f^\times \circ -"] \\
Ri(\mathbb{Z}G, R') \arrow[r, "\sim"'] & Gps(G, (R')^\times)
\end{tikzcd}$$

commutes.

α

$\downarrow$

$f\circ\alpha$

$(\llcorner)$: $f\circ\alpha \longmapsto \phi(f\circ\alpha) : G \to (R')^\times$

$g \mapsto f\circ\alpha(1g + \dots)$ — check that this is a unit in R'

$\underset{\mathbb{Z}G}{(1g+0+0\dots)} \mapsto \underset{R'}{f\circ\alpha(1g+0+\dots)}$ ~~$f\circ\alpha(g)$~~

$(\urcorner)$: $\alpha \longmapsto \phi(\alpha) : G \to R^\times$

$g \mapsto \phi(\alpha)(g)$

$\downarrow$

$f^\times \circ \varphi(\alpha) : G \to (R')^\times$

$g \mapsto f^\times(\varphi(\alpha)(g)) = f(\varphi(\alpha)(g))$

$=?$

What the heck is $f^\times$?

Given $R \xrightarrow{f} R'$ a ring hom.

$$\begin{array}{ccc} R^\times & \xrightarrow{f^\times} & (R')^\times \\ \rotatebox{90}{$\in$} & & \\ r & \longmapsto & f(r) \\ & & \| \\ & & f^\times(r) \end{array}$$

a gp. hom. btwn. unit gps.

Check that a ring. hom. sends units $\longmapsto$ units.