10/6 Twoples meeting.

· Homotopy limits/colimits.

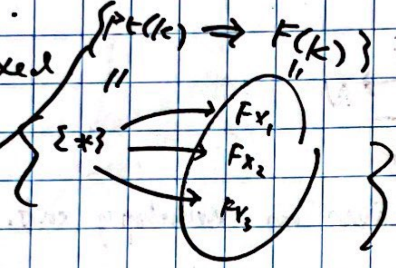
$F: K \longrightarrow \mathcal{C}$

$[K, Set]$ — obs = functors $K \to Set$.
— maps = nat'l transf's.

$pt: K \longrightarrow Set$
$k \longmapsto \{*\}$.

$F: K \to Set$. fixed

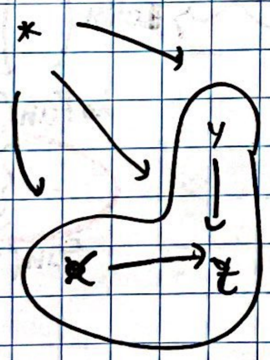
$\mathrm{Hom}_{[K,Set]}(pt, F)$



E.g. $K = \{\bullet \to \bullet \leftarrow \bullet\}$.

$F: K \to Set$ $\quad\quad$ $Nat(pt, F)$

$$\begin{array}{ccc} & & y \\ & & \downarrow \\ x & \to & z \end{array}$$



$$\mathcal{C} \quad \mathrm{colim} \left(\dashv \Big\downarrow \text{const.} \right) \lim \quad \mathcal{C}^K$$

~~Nat(lim F, W) ≅ Hom(lim F, ...)~~

$\mathrm{Hom}_{\mathcal{C}}(\mathrm{colim}_K F, W) \cong Nat(F, \mathrm{const}_W)$.

$\mathrm{Hom}_{Set}(\{*\}, \lim F) \cong$
$\lim F$

I-shaped
~~Say~~ $\mathcal{C}$ has all limits | colimits..
Then there exist adjunctions ...

$$\begin{tikzcd} \mathcal{C} \arrow[d, "\mathrm{const.}"] \\ \mathrm{Fun}(I,\mathcal{C}) \arrow[u, bend left, "\mathrm{colim}"] \arrow[u, bend right, "\lim"'] \end{tikzcd}$$

~~Recall: ...~~

Now if ~~[crossed out]~~ M, N a model cats ~~w/ all I-shaped lims/colims..~~
~~we can describe~~
w/ an adjunction $M \underset{G}{\overset{F}{\rightleftarrows}} N$
(...?)

$$\begin{tikzcd} M \arrow[r, "F"] \arrow[d, "\gamma"] & N \arrow[d, "\gamma"] \\ ho(M) \arrow[r, dashed, "\mathbb{L}F"] & ho(N) \end{tikzcd}$$

then there form an adjunction on homotopy cats.

$$ho(M) \underset{RG}{\overset{LF}{\rightleftarrows}} ho(N)$$

You can describe homotopy lims/colims as derived functors..
ie.

$$\begin{tikzcd} \mathcal{C} \arrow[d, "\mathrm{const.}"] \\ \mathrm{Fun}(I,\mathcal{C}) \arrow[u, bend left, "\mathrm{colim}"] \end{tikzcd} \quad \overset{\text{derived}}{\rightsquigarrow} \quad \begin{tikzcd} ho(\mathcal{C}) \arrow[d, "R\mathrm{const.}"] \\ ho\mathrm{Fun}(I,\mathcal{C}) \arrow[u, bend left, "L\mathrm{colim}"] \arrow[d, leftrightsquigarrow, "??"] \\ \mathrm{Fun}(I, ho(\mathcal{C})) \end{tikzcd}$$

model cat.

?

Q: when is a functor cat. into a model cat. a model cat.?
"cofibrantly generated"
"model structure on functor cats"

$$\begin{tikzcd} ho(\mathcal{C}) \arrow[d, "\mathrm{const.}"] \\ \mathrm{Fun}(I, ho(\mathcal{C})) \arrow[u, bend left, "\begin{array}{c} L\mathrm{colim} \\ \shortparallel \\ \mathrm{hocolim} \end{array}"] \end{tikzcd}$$

· simplicial model cats
· combinatorial model cats.

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[dr, "K"'] & E \\ & D \end{tikzcd}$$

$K \downarrow d$

$\shortparallel$

$K_{/d}$ — ob = { maps $\left(\begin{smallmatrix} x \\ \downarrow \\ d \end{smallmatrix}\right)$ in D }.

mor = $\left\{ \begin{tikzcd}[sep=small] x \arrow[r] \arrow[dr] & y \arrow[d] \\ & d \end{tikzcd} \right\}$.

$K \downarrow d$ = cat. of elts of $D(K-, d) : C^{op} \to set.$

$$K_{/d} \xrightarrow{proj} K$$

$\left(\begin{smallmatrix} x \\ \downarrow \\ d \end{smallmatrix}\right) \longmapsto x$

$\left(\begin{tikzcd}[sep=small] x \arrow[r] \arrow[dr] & y \arrow[d] \\ & d \end{tikzcd}\right) \longmapsto (x \to y)$

$\mathcal{C}$: an ordinary cat.

$\rightsquigarrow$

$sSet \ni N\mathcal{C}$ — $N\mathcal{C}_0 = ob(\mathcal{C})$

qCat $\ni$

$N\mathcal{C}_1 = Hom(\mathcal{C})$

$N\mathcal{C}_2$ = {pairs $\bullet \xrightarrow{f_1} \bullet \xrightarrow{f_2} \bullet$ in $\mathcal{C}$}.

$\vdots$

$N\mathcal{C}_n$ = {strings $\bullet \xrightarrow{f_1} \bullet \xrightarrow{f_2} \bullet \to \cdots \xrightarrow{f_n} \bullet$ in $\mathcal{C}$}.

$\mathcal{C}$: a 2-cat...

$$\begin{tikzcd}[sep=small] & \bullet \arrow[dr, "f_2"] & \\ \bullet \arrow[ur, "f_1"] \arrow[rr] \arrow[rr, phantom, shift left=3, "\Downarrow"] & & \bullet \end{tikzcd} \quad \ldots \text{ or } \quad \begin{tikzcd}[sep=small] & \bullet \arrow[dr, "f_2"] & \\ \bullet \arrow[ur, "f_1"] \arrow[rr] \arrow[rr, phantom, shift left=3, "="] & & \bullet \end{tikzcd}$$

strict vs. nonstrict?

$N_2\mathcal{C}$ — ob $= \mathcal{C}_0$

1-mor $= \mathcal{C}_1$

2-mor = $\exists\ \Delta^2 \to N_2\mathcal{C}$ for any 2-mor.

$$\mathcal{C}_2 = \left\{ \begin{tikzcd} \bullet \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd} \right\}.$$

$$\begin{tikzcd}[sep=small] & \bullet \arrow[dr, equal, "id"] & \\ \bullet \arrow[ur, "f"] \arrow[rr, "g"'] \arrow[rr, phantom, shift left=3, "\Downarrow"] & & \bullet \end{tikzcd} \qquad \begin{tikzcd}[sep=small] & \bullet \arrow[dr, "f"] & \\ \bullet \arrow[ur, equal] \arrow[rr, "g"'] \arrow[rr, phantom, shift left=3, "\Downarrow"] & & \bullet \end{tikzcd}$$

$\vdots$

Q: do we need more?.

Q: How do we describe compositions of 1-morphisms?

Q: Can we describe horizontal/vertical comp. of 2-morphisms as horn-filling in $N_2\mathcal{C}$

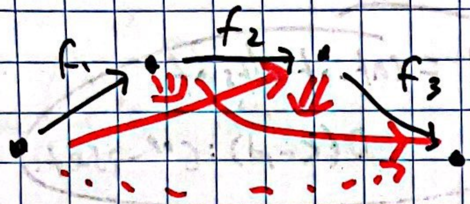
$$\begin{tikzcd} \Lambda \arrow[r] \arrow[d, hook] & N_2\mathcal{C} \\ \Delta \arrow[ur, dotted, "?"'] & \end{tikzcd}$$

Q: How do we know if we have a suitable $N_2\mathcal{C}$?.

A: should be able to recover the 2-cat. $\mathcal{C}$ by taking h_2 (htpy 2-cat.)

ie. $\exists$ an equiv. of 2-cats. $\mathcal{C} \cong h_2 N_2 \mathcal{C}$

In a 2-cat... Say you have 3 1-morphisms...



- How do we describe these 2 & 3-morphisms in the nerve?.
- How do we know 2-morphisms fit together in a coherent way?.