Twoples.
9/21/22.

1. homotopy theory $\longleftrightarrow$ homology. —— derived functors – Ext, Tor.
— What is "abelianness"?
—

· model & ∞-cats.
— read: Groth
Hovey.
Dwyer-Spalinski

Pick a comm. ring R. $\rightsquigarrow$ R-Mod $\rightsquigarrow$ Ch(R)

additive ab. abelian cat. — modules over R — ch. cpxes of R-modules

Top $\overset{?}{\longleftrightarrow}$ Ch(R)
Noori – Triangulated cats?

Q: What should be weak eq's?
(A: homology invariant.)

· chain homotopy

Given ch. cpxes $C_\bullet, D_\bullet \in Ch(R)$

$$\begin{tikzcd} C_\bullet \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow h"] & D_\bullet \end{tikzcd} \quad \{h_n : C_n \to D_{n+1} \mid \cdots\}.$$

$$\begin{tikzcd} \cdots \arrow[r] & C_n \arrow[r, "d"] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] & C_{n+1} \arrow[r] \arrow[dl, "h_{n+1}"] & C_{n+2} \arrow[r] \arrow[dl, "h_{n+2}"] & \cdots \\ \cdots \arrow[r] & D_n \arrow[r, "d"'] & D_{n+1} \arrow[r] & D_{n+2} \arrow[r] & \cdots \end{tikzcd}$$

Define "homotopy category of ch. cpxes"

$K(R)$ — ob = {ch. cpxes}.
mor: $\mathrm{Hom}_{K(R)}(C_\bullet, D_\bullet) = \mathrm{Hom}_{Ch(R)}(C_\bullet, D_\bullet) / \sim$ ch. htpy.

$Ch(R) \longrightarrow K(R)$

But the notion of chain homotopy turns out not to be the one we want — !.
(in some ways it does, e.g. if ($f \underset{\text{ch. htpy}}{\sim} g$) $\Rightarrow$ $\bar{f} = \bar{g}$ are the same on homology.)

Turns out that the "right" notion of htpy is $\boxed{\text{quasi-isomorphism}}$

Def. ~~maps $f, g : C_\bullet \rightrightarrows D_\bullet$~~ ~~$f \underset{q.iso.}{\sim} g$ iff~~ A map $f : C_\bullet \longrightarrow D_\bullet$ is a q. iso. if the induced maps $f_n : H_n(C_\bullet) \to H_n(D_\bullet)$ are isomorphisms (of abelian gps).

$\rightsquigarrow$ describe a cat.

(!)

This turns out to be the "right" notion of weak eq. of ch. cpxes.

⇝ can describe $D(R)$ — ob = ch. cpxes.

$\mathrm{Hom}_{D(R)}(C_\bullet, D_\bullet) = \mathrm{Hom}_{Ch(R)}(C_\bullet, D_\bullet) / \sim$ q.iso.

derived cat. of a ring (ab. cat.)

$$\begin{tikzcd} Ch(R) \arrow[r] \arrow[rr, bend right] & K(R) \arrow[r] & D(R) \end{tikzcd}$$

htpy cat. "naive htpy cat."

derived cat. ⟷ derived functors. "true htpy cat".

- Balchin
- Hovey
- Quillen - Homotopical Algebra

HA Ch.1 $\mathcal{D}(R)$: ∞-"derived cat.

- combinatorial model cats. ⇝ (locally) presentable ∞-categories.
  - "Jeff Smith's theorem"
  - "generated by "small" data" i.e. "easy" to get a handle on.

~~accessibly~~ accessible.

model cat ⇝ "underlying ∞-cat."

simplicial localization / Dwyer-Kan

$\mathcal{C}$: ∞-cat. ⇝ $\mathcal{C}_1$: a 1-cat.

$\mathcal{C}_2$: a 2-cat.

⋮

— $h_2\mathcal{C}$

— $h_3\mathcal{C}$

Riehl-Verity "Elements of ∞-cat. theory" Ch1.

$$\begin{tikzcd} & Y \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] & {} & Z \end{tikzcd}$$

⇝ in $\mathcal{C}_1$ should this be a commutative

$$\begin{tikzcd} & Y \arrow[dr, "g"] \arrow[d, phantom, "="] & \\ X \arrow[ur, "f"] \arrow[rr, "h = g\circ f"'] & {} & Z \end{tikzcd}$$

... $h\mathcal{C}$

or

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "g\circ f"] \arrow[rr, bend right=50, "h"'] & {} & Z \end{tikzcd}$$

$\mathcal{C}$: ∞-cat.

$h_3\mathcal{C}$ — ob = $\mathcal{C}_0$
mor = $\mathcal{C}_1$
2-mor = $\mathcal{C}_2$
3-mor = $\mathcal{C}_3/\sim$ higher htpies.

vs. $h_3^{en.}\mathcal{C}$ — ?? describe as a cat. enriched over (strict?/weak?) 2-cats.

ex/1.

- Look up $h_2\mathcal{C}$ in ~~[illegible]~~ Riehl-Verity. (weak?/strict?)
- ~~Define a~~ Describe it as a cat. enriched ~~over cat~~ over a 1-cat..
- Show the equivalence (of 2-cats)

∞-cats. — ~~ssets~~ (quasi-categories)
(n-morphisms for all n)

(as simplicially enriched cats.
i.e. categories whose hom-sets are ssets)

(as dg-cats.
i.e. cats who are enriched in diff. graded ab-gps)

in HA to construct $\mathcal{D}(A)$ ———— ∞-cat.

$h\mathcal{D}(A)$ ———— 1-cat.
$\shortparallel$
$D(A)$

To do : · study the model cat. of chcpxes ~~of modules of~~ ...
· derived vs. homot. of an abelian cat. A
$D(A)$ $K(A)$
· relative categories as a model of ∞-cat.
· quasi-categories (Kerodon)
· $h_2\mathcal{C}$ the htpy 2-cat as an enriched cat ($h_3\mathcal{C}$)